RIGIDITY IN THE LANGLANDS CORRESPONDENCE AND APPLICATIONS

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Contents

1. Introduction

In this report, we give a survey on some new construction of local systems on open curves. The construction uses the idea of rigid automorphic representations which we will try to axiomatize here. We also give applications of these constructions to more classical problems in algebraic geometry and number theory. The main references are [17] and [36]. More details on rigid automorphic representations will appear in the article [38].

1.1. **The goal.** Let us fix an algebraic curve X over an algebraically closed field k. Let $S \subset X$ be a finite set of closed points. Let ℓ be a prime different from $\mathrm{char}(k)$. A local system $\mathcal{F}$ (with $\overline{\mathbb{Q}}_\ell$-coefficients, in the étale topology) over $U := X - S$ is *physically rigid* if it is determined up to isomorphism by its local monodromy around points $x \in S$. A local system $\mathcal{F}$ over U is *cohomologically rigid* if $\mathrm{H}^1(X, j_*\mathrm{End}^\circ(\mathcal{F})) = 0$, where $\mathrm{End}^\circ(\mathcal{F})$ is the local system of trace-free endomorphisms of $\mathcal{F}$, and j_* means the sheaf (not derived) push-forward along $j : X - S \to X$. These notions were defined and studied in depth by N. Katz in [20]. The main result of [20] is an algorithmic description of tame local systems.

We shall be interested in local systems in a broader sense. Let H be a connected reductive algebraic group over $\overline{\mathbb{Q}}_\ell$. An H-local system on U is a continuous homomorphism $\rho : \pi_1(U) \to H(\overline{\mathbb{Q}}_\ell)$, where $\pi_1(U)$ is the étale fundamental group of U. This specializes to the notion of rank n local systems when $H = \mathrm{GL}_n$. Both notions of rigidity can easily be extended to H-local systems.

We would like to construct many examples of rigid H-local systems, especially when H is of exceptional types. The key observation is the following: using the Langlands correspondence for function fields, it is easier to construct the automorphic counterpart of these rigid local systems than constructing the local systems themselves.

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1.2. **The method.** Suppose for the moment that the complete algebraic curve X is defined over a finite field k. Let G be a split reductive group k. Langlands philosophy predicts that there should be a finite-to-one correspondence from automorphic representations π of $G(\mathbb{A}_F)$ (where $\mathbb{A}_F$ is the ring of adèles of F) to $\widehat{G}$-local system ρ_π over an open subset of X. Here $\widehat{G}$ is a group over $\overline{\mathbb{Q}}_\ell$ whose root system is dual to that of G. When $G = \mathrm{GL}_n$, this correspondence is proved by L.Lafforgue [21], and it is in fact one-to-one when restricted to cuspidal automorphic representations. For general connected reductive G, the assignment $\pi \mapsto \rho_\pi$ has recently be established by V.Lafforgue [22]. However, the construction of ρ_π is not easy to carry out explicitly in concrete examples.

We shall propose a notion of a rigid automorphic datum with respect to a given finite set of closed points $S \subset X$. Here is the simplest version of it. For each $x \in S$ one imposes a condition on the local component π_x of an automorphic representation π. This set of local conditions is called a rigid automorphic datum if there is up to isomorphism only one automorphic representation π which is unramified outside S and satisfies the given local conditions at each $x \in S$. In practice, it is easy to write down a Hecke eigenform in π as a function f on the double coset $G(F)\backslash G(\mathbb{A}_F)/K$ (for a suitable compact open subgroup $K \subset G(\mathbb{A}_F)$).

Once we have the Hecke eigenform f in π, via the Satake isomorphism, the Hecke eigenvalues of f at a place $x \notin S$ determine the image of the Frobenius element Frob_x under ρ_π. Although this much information sometimes determines the isomorphism type of ρ_π, we still do not have the local system ρ_π as a concrete geometric object. To get ρ_π geometrically, we appeal to the geometric Langlands correspondence, which is a sheaf-theoretic upgrading of the Langlands correspondence for function fields. As suggested by an observation of Weil, one should think of Bun_G, the moduli space of G-bundles over X (with suitable level structure), as a geometric incarnation of the double coset $G(F)\backslash G(\mathbb{A}_F)/K$. In the setting of geometric Langlands, one should try to upgrade the Hecke eigenform f to a sheaf (or complexes of sheaves) $\mathcal{F}$ on Bun_G. The precise meaning of "upgrade" is encoded in the "sheaf-to-function" correspondence of Grothendieck. Once the Hecke eigensheaf $\mathcal{F}$ is constructed, we can construct the corresponding $\widehat{G}$-local system ρ_π in a completely explicit manner, via geometric Hecke operators. Of course in general it is not an easy task to construct Hecke eigensheaves $\mathcal{F}$. However, when π satisfies a rigid automorphic datum, there is essentially only one choice for $\mathcal{F}$, which forces it to be a Hecke eigensheaf.

To summarize, we start from a rigid automorphic datum for $G(\mathbb{A}_F)$, find the unique automorphic representation π satisfying this rigid automorphic datum and a Hecke eigenform f in it, then we use geometric Langlands to upgrade f into a sheaf $\mathcal{F}$, and finally we apply geometric Hecke operators to $\mathcal{F}$ to get a $\widehat{G}$-local system ρ_π.

1.3. **Applications.** We shall give three applications of the rigid objects in the Langlands correspondence.

1.3.1. *Local systems with exceptional monodromy groups.* Katz [19] has constructed an example of a local system over $\mathbb{P}^1_{\mathbb{F}_p} - \{0, \infty\}$ whose geometric monodromy lies in the exceptional group G_2 and is Zariski dense there. This G_2-local system comes from a rank 7 rigid local system which is an example of Katz's hypergeometric sheaves. For other exceptional groups, there weren't explicitly constructed examples of local systems with Zariski dense monodromy in them.

In joint work with Heinloth and Ngô [17], for any reductive group $\widehat{G}$ over $\overline{\mathbb{Q}}_\ell$, we constructed a $\widehat{G}$-local system $\mathrm{Kl}_{\widehat{G}}$ over $\mathbb{P}^1_{\mathbb{F}_p} - \{0, \infty\}$. The geometric monodromy of these local systems is quite large. For example, when $\widehat{G}$ is of type E_7, E_8, F_4 or G_2, the monodromy is Zariski dense in $\widehat{G}$. When $\widehat{G} = \mathrm{GL}_n$ or Sp_n, $\mathrm{Kl}_{\widehat{G}}$ is the same as the Kloosterman sheaf constructed by Deligne [5];

when $\widehat{G} = G_2$, $\mathrm{Kl}_{\widehat{G}}$ is the same as the G_2 example of Katz. These generalized Kloosterman sheaves $\mathrm{Kl}_{\widehat{G}}$ give first examples of motivic local systems with Zariski dense monodromy in exceptional groups other than G_2. The associated Frobenius traces give new exponential sums generalizing the Kloosterman sums, and they are equidistributed according to the Sato-Tate measure for compact Lie groups of type E_7, E_8, F_4 or G_2.

Our construction in [17] was inspired by a result of B.Gross [13] and the work of Frenkel and Gross [10] on rigid irregular connections. Gross showed in [13] that there exists a unique automorphic representation of G over the rational function field $F = k(t)$ whose local component at 0 is a Steinberg representation and at ∞ is a simple supercuspidal representation. He then conjectured that when $G = \mathrm{GL}_n$, the Satake parameters of this automorphic representation should give the classical Kloosterman sums. Our work [17] confirmed this conjecture and generalized it to other reductive groups.

1.3.2. *Motives over number fields with exceptional motivic Galois groups.* In early 1990s, Serre asked the following question [32]: Is there a motive over a number field whose motivic Galois group is of exceptional type such as G_2 or E_8?

A motive M over a number field K is, roughly speaking, part of the cohomology $\mathrm{H}^i(X)$ for some (smooth projective) algebraic variety X over K and some integer i, which is cut out by geometric operations (such as group actions). For each prime ℓ, the motive M has the associated ℓ-adic cohomology $\mathrm{H}_\ell(M) \subset \mathrm{H}^i(X_{\overline{K}}, \overline{\mathbb{Q}}_\ell)$, which admits a Galois action:

$$\rho_{M,\ell} : \mathrm{Gal}(\overline{K}/K) \to \mathrm{GL}(\mathrm{H}_\ell(M))$$

The *ℓ-adic motivic Galois group* $G_{M,\ell}$ of M is the Zariski closure of the image of $\rho_{M,\ell}$. This is an algebraic group over $\overline{\mathbb{Q}}_\ell$. Since the motivic Galois groups that appear in the original question of Serre are only well-defined assuming the standard conjectures in algebraic geometry, we shall use the ℓ-adic motivic Galois group as a working substitute for the actual motivic Galois group (conjecturally they should be isomorphic to each other). Classical groups appear as motivic Galois groups of abelian varieties. This is why Serre raised the question for exceptional groups only. Until recently, the only known case of Serre's question was G_2, by the work of Dettweiler and Reiter [8].

In [36], we construct motivic local systems on $\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$ with Zariski dense monodromy in exceptional groups E_7, E_8 and G_2 in a uniform way. As a consequence of this construction, we give an affirmative answer to the ℓ-adic version of Serre's question for E_7, E_8 and G_2: these groups can be realized as the ℓ-adic motivic Galois groups for motives over number fields (in fact the number field is either $\mathbb{Q}$ in the case of E_8 and G_2, or $\mathbb{Q}(i)$ in the case of E_7). With a bit more work, one can also realize F_4 as a motivic Galois group over $\mathbb{Q}$.

We remark that naive attempts to finding motives with motivic Galois group of type E_8 can be quite painful, if not hopeless. Usually people look for motives from Shimura varieties or abelian varieties. However, it is known that abelian varieties do not have exceptional motivic Galois groups; nor is there a Shimura variety of type E_8. To find an E_8-motive, at the very least, one needs to find an algebraic variety over a number field K whose cohomology in some degree has dimension at least 248 (because the smallest nontrivial representation of E_8 has dimension 248, the adjoint representation). Our construction gives a 248-dimensional Galois representation in a natural way from the geometry of the algebraic group E_8 itself.

1.3.3. *Inverse Galois Problem.* The inverse Galois problem over $\mathbb{Q}$ asks whether every finite group can be realized as the Galois group of some Galois extension $K/\mathbb{Q}$. The problem is still open for many finite simple groups, especially those of Lie type. The same rigid local systems over

$\mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$ constructed to answer Serre's question can be used to solve new cases of the inverse Galois problem. We show in [36] that for sufficiently large primes ℓ, the finite simple groups $G_2(\mathbb{F}_\ell)$ and $E_8(\mathbb{F}_\ell)$ can be realized as Galois groups over $\mathbb{Q}$. With a bit more work, one can show that $F_4(\mathbb{F}_\ell)$ is also a Galois group over $\mathbb{Q}$.

In inverse Galois theory, people use the "rigidity method" to prove certain finite groups H are Galois groups over $\mathbb{Q}$. In particular, the case of $G_2(\mathbb{F}_\ell)$ for all primes $\ell \geq 5$ was known by the work of Thompson [35] and Feit and Fong [9]. However, the case of $E_8(\mathbb{F}_\ell)$ was known only for primes ℓ satisfying a certain congruence condition modulo 31 (see the book of Malle and Matzat [27, II.10] for a summary of what was known before). Therefore our result for $E_8(\mathbb{F}_\ell)$ for all ℓ sufficiently large is new.

The input data of the rigid method in inverse Galois theory is a triple of conjugacy classes in the target group H. The notion of rigidity of such a triple is quite similar to the notion of physically rigidity for local systems over $\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}$. However, connection between the rigidity method and automorphic representations has not been explored before. Our result shows that this connection can be useful in solving the inverse Galois problem, and it even sheds some light to the rigidity method itself. In fact, our construction of the local system over $\mathbb{P}^1 - \{0, 1, \infty\}$ suggests a triple in $E_8(\mathbb{F}_\ell)$ which should be a rigid triple (see [36, Conjecture 5.16]). This has been confirmed by Guralnick and Malle [16], and using this triple they are able to show that $E_8(\mathbb{F}_\ell)$ is a Galois group over $\mathbb{Q}$ for all primes $\ell > 7$.

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2. Rigidity for automorphic representations

In this section we recall some basic concepts for automorphic representations over a function field. We will define the notion of a geometric automorphic datum (a collection of local conditions for automorphic representations to satisfy), and define what it means for such a datum to be rigid.

2.1. **The setting.**

2.1.1. *Function field.* Let k be a finite field. Let X be a projective, smooth and geometrically connected curve X over k. Let $F = k(X)$ be the field of rational functions on X.

Let $|X|$ be the set of closed points of X (places of F). For each $x \in |X|$, let F_x denote the completion of F at the place x. The valuation ring and residue field of F_x are denoted by $\mathcal{O}_x$ and k_x. The maximal ideal of $\mathcal{O}_x$ is denoted by $\mathfrak{m}_x$. The ring of adèles is the restricted product

$$\mathbb{A}_F := \prod_{x \in |X|}{}' F_x$$

where for almost all x, the x-component of an element $a \in \mathbb{A}_F$ lies in $\mathcal{O}_x$. There is a natural topology on $\mathbb{A}_F$ making it a locally compact topological ring.

2.1.2. *Groups.* Although most of the results in this report are valid for connected quasisplit reductive groups over F, but for the simplicity of the presentation, we restrict ourselves to the following situation: G **is a connected, split, simply-connected and almost simple group over** k. We will fix a pinning of G, namely a split maximal torus T, a Borel subgroup $B \supset T$ and generators of simple roots spaces. It makes sense to talk about $G(R)$ whenever R is a k-algebra. For example, $G(F)$, $G(\mathcal{O}_x)$ and $G(F_x)$, etc.

Later we shall also need to view $G(F_x)$ and $G(\mathcal{O}_x)$ as infinite-dimensional groups over k. More precisely, for a k_x-algebra R we let $L_xG(R) = G(R\widehat{\otimes}_{k_x}F_x)$ and let $L_x^+G(R) = G(R\widehat{\otimes}_{k_x}\mathcal{O}_x)$. Here $R\widehat{\otimes}_{k_x}F_x$ or $R\widehat{\otimes}_{k_x}\mathcal{O}_x$ means the completion of the tensor product with respect to the $\mathfrak{m}_x$-adic topology on F_x or $\mathcal{O}_x$. The functor L_xG (resp. L_x^+G) is representable by a group indscheme (resp. group scheme of infinite type) over k_x. There is a reduction map $L_x^+G \to G \otimes_k k_x$.

We will also need the notion of *parahoric subgroups* of $G(F_x)$. Bruhat and Tits [4] constructed certain smooth models $\mathcal{P}$ of G over $\mathcal{O}_x$ whose special fibers are connected. By a parahoric subgroup $\mathbf{P}$ of L_xG we shall mean the group scheme over k_x whose R-point is $\mathcal{P}(R\widehat{\otimes}_{k_x}\mathcal{O}_x)$ where $\mathcal{P}$ is a Bruhat-Tits group scheme over $\mathcal{O}_x$. The k_x-points $\mathbf{P}(k_x)$ of $\mathbf{P}$ is a parahoric subgroup of $G(F_x)$ in the usual sense. Viewed as a proalgebraic group, the parahoric $\mathbf{P} \subset L_xG$ has a maximal reductive quotient $L_{\mathbf{P}}$ which is a connected reductive group over k_x. For example, the preimage $\mathbf{I}_x$ of $B \otimes_k k_x$ under the reduction map $L_x^+G \to G \otimes_k k_x$ is called the *standard Iwahori subgroup* of L_xG (with respect to B), and any parahoric subgroup of L_xG contains a conjugate of $\mathbf{I}_x$.

Finally, the Langlands dual group $\widehat{G}$ to G is a reductive group over $\overline{\mathbb{Q}}_\ell$, equipped with a pinning $(\widehat{T}, \widehat{B}, \cdots)$ such that the corresponding based root system $(\mathbb{X}^*(\widehat{T}), \Phi(\widehat{G}, \widehat{T}), \Delta_{\widehat{B}})$ of $\widehat{G}$ is the same as the based coroot system $(\mathbb{X}_*(T), \Phi^\vee(G,T), \Delta_B^\vee)$ of G with respect to T and B.

2.1.3. *Automorphic representations.* The group $G(\mathbb{A}_F)$ of $\mathbb{A}_F$-points of G can also be expressed as the restricted product

$$G(\mathbb{A}_F) = \prod_{x \in |X|}{}' G(F_x)$$

where most components lie in $G(\mathcal{O}_x)$. This is a locally compact topological group. With this topology on $G(\mathbb{A}_F)$ we may talk about locally constant functions on $G(\mathbb{A}_F)$. Denote the space of locally constant $\overline{\mathbb{Q}}_\ell$-valued functions on $G(\mathbb{A}_F)$ by $C^\infty(G(\mathbb{A}_F), \overline{\mathbb{Q}}_\ell)$. Define $C^\infty(G(F)\backslash G(\mathbb{A}_F), \overline{\mathbb{Q}}_\ell)$ as the subspace of $C^\infty(G(\mathbb{A}_F), \overline{\mathbb{Q}}_\ell)$ consisting of left-$G(F)$-invariant functions.

2.1.4. **Definition** (See [3, Definition 5.8]).
(1) A function $f \in C^\infty(G(F)\backslash G(\mathbb{A}_F), \overline{\mathbb{Q}}_\ell)$ is called an *automorphic form* if for some (equivalently any) $x \in |X|$, the $G(F_x)$-module spanned by right $G(F_x)$-translations on f is admissible. Denote the space of automorphic forms by $\mathcal{A}_G$. This is a $G(\mathbb{A}_F)$-module under right translation.
(2) An irreducible representation π of $G(\mathbb{A}_F)$ is an *automorphic representation* if it is a subquotient of $\mathcal{A}_G$.

2.1.5. *Cusp forms.* A function $f \in C^\infty(G(F)\backslash G(\mathbb{A}_F), \overline{\mathbb{Q}}_\ell)$ is a *cusp form* if for every parabolic subgroup $P \subset G$ defined over F with unipotent radical N_P, we have

$$\int_{N_P(F)\backslash N_P(\mathbb{A}_F)} f(ng)dn = 0$$

for all $g \in G(\mathbb{A}_F)$. All cusp forms have compact support in $G(F)\backslash G(\mathbb{A}_F)$. The space of cusp forms is a sub-$G(\mathbb{A}_F)$-module $\mathcal{A}_G^{\mathrm{cusp}}$ of all automorphic forms $\mathcal{A}_G$. It is known that $\mathcal{A}_G^{\mathrm{cusp}}$ decomposes discretely into a direct sum of irreducible $G(\mathbb{A}_F)$-modules, called *cuspidal automorphic representations.*

2.2. **Weil's interpretation.** We temporarily allow G to be a reductive group over k. When $G = \mathrm{GL}_n$, Weil has given a geometric interpretation of the double coset $G(F)\backslash G(\mathbb{A}_F)/\prod_{x\in|X|} G(\mathcal{O}_x)$ in terms of vector bundles over X. We recall his interpretation for general reductive G.

A (right) G-torsor over X is a scheme $Y \to X$ together with a fiberwise action of G that looks like $G \times X$ (with G acting on itself by right translation) étale locally over X. An isomorphism between G-torsors Y and Y' is a G-equivariant isomorphism $Y \cong Y'$ over X.

Let $\mathrm{Bun}_G(k)$ be the groupoid of G-torsors over X: this is a category whose objects are G-torsors over X and morphisms are isomorphisms between G-torsors. The groupoid $\mathrm{Bun}_G(k)$ is in fact the groupoid of k-points of an algebraic stack Bun_G. For a k-algebra R the groupoid $\mathrm{Bun}_G(R)$ is the groupoid of G-torsors over $X \otimes_k R$.

2.2.1. **Example.** When $G = \mathrm{GL}_n$, there is a natural way to assign a vector bundle $\mathcal{V}$ of rank n over X to a GL_n-torsor $\mathcal{E}$ and vice versa: $\mathcal{V} = \mathcal{E} \overset{\mathrm{GL}_n}{\times} \mathbb{A}^n_k$ and $\mathcal{E} = \underline{\mathrm{Isom}}_X(\mathcal{O}_X^n, \mathcal{V})$. Therefore $\mathrm{Bun}_{\mathrm{GL}_n}$ is isomorphic to the algebraic stack Bun_n classifying rank n vector bundles over X. In particular, for $n = 1$, $\mathrm{Bun}_{\mathrm{GL}_1} \cong \mathrm{Bun}_1 \cong \mathrm{Pic}(X)$.

Similarly, $\mathrm{Bun}_{\mathrm{SL}_n}$ classifies pairs $(\mathcal{V}, \iota)$ where $\mathcal{V}$ is a vector bundle of rank n over X and $\iota : \wedge^n \mathcal{V} \xrightarrow{\sim} \mathcal{O}_X$ is a trivialization of the determinant of $\mathcal{V}$.

When $G = \mathrm{PGL}_n$, $\mathrm{Bun}_{\mathrm{PGL}_n}$ is equivalent to the quotient stack $\mathrm{Bun}_n/\mathrm{Pic}(X)$ with $\mathrm{Pic}(X)$ acting on Bun_n via tensor product. More concretely, the objects in $\mathrm{Bun}_{\mathrm{PGL}_n}(k)$ are the same as the objects in $\mathrm{Bun}_n(k)$, and the morphism set between two vector bundles $\mathcal{V}$ and $\mathcal{V}'$ in the groupoid $\mathrm{Bun}_{\mathrm{PGL}_n}(k)$ is the set of isomorphism classes of line bundles $\mathcal{L}$ on X such that $\mathcal{V} \otimes \mathcal{L} \cong \mathcal{V}'$. Here we have used the fact that the Brauer group of X is trivial, so that all PGL_n-torsors over X lift to GL_n-torsors.

Weil observed that there is a natural bijection

$$e : G(F)\backslash G(\mathbb{A}_F)/\prod_{x \in |X|} G(\mathcal{O}_x) \xrightarrow{\sim} \mathrm{Bun}_G(k). \tag{2.1}$$

This is not just a bijection of sets, but in fact an isomorphism of groupoids. In other words, for any double coset $[g] = G(F) g \prod_{x \in |X|} G(\mathcal{O}_x)$, the automorphism group of $e([g])$ (as a G-torsor) is isomorphic to the stabilizer of the coset $g \prod_{x \in |X|} G(\mathcal{O}_x)$ under $G(F)$.

We give the definition of the map e on the level of sets. For any finite $S \subset |X|$, the double coset $G(\mathcal{O}_{X-S})\backslash \prod_{x \in S} G(F_x)/G(\mathcal{O}_x)$ is the subset of $G(F)\backslash G(\mathbb{A}_F)/\prod_{x \in |X|} G(\mathcal{O}_x)$ consisting of the those classes represented by $g = (g_x) \in G(\mathbb{A}_F)$ with $g_x \in G(\mathcal{O}_x), \forall x \notin S$. Clearly,

$$G(F)\backslash G(\mathbb{A}_F)/\prod_{x \in |X|} G(\mathcal{O}_x) = \bigcup_{S \subset |X|, \text{finite}} G(\mathcal{O}_{X-S})\backslash \prod_{x \in S} G(F_x)/G(\mathcal{O}_x).$$

Thus it suffices to construct a compatible system of maps

$$e_S : G(\mathcal{O}_{X-S})\backslash \prod_{x \in S} G(F_x)/G(\mathcal{O}_x) \to \mathrm{Bun}_G(k).$$

For $g = (g_x) \in \prod_{x \in S} G(F_x)$, we take the trivial G-torsor $\mathcal{E}^{\mathrm{triv}}_{X-S} = (X - S) \times_k G$ over $X - S$ and glue it with the trivial G-torsor $\mathcal{E}^{\mathrm{triv}}_x = \mathrm{Spec}\mathcal{O}_x \times G$ over $\mathrm{Spec}\mathcal{O}_x$ for each $x \in S$ along $\mathrm{Spec}F_x$. The gluing means to give an isomorphism $\mathcal{E}^{\mathrm{triv}}_x|_{\mathrm{Spec}F_x} \xrightarrow{\sim} \mathcal{E}^{\mathrm{triv}}_{X-S}|_{\mathrm{Spec}F_x}$, which is given by g_x. Changing the trivializations of $\mathcal{E}^{\mathrm{triv}}_{X-S}$ and $\mathcal{E}^{\mathrm{triv}}_x$ amounts to right multiply g_x by an element in $G(\mathcal{O}_x)$ and left multiply g_x by an element in $G(\mathcal{O}_{X-S})$. The isomorphism type of the resulting G-torsor $\mathcal{E}_g$ after gluing only depends on the double coset $G(\mathcal{O}_{X-S}) g \prod_{x \in S} G(\mathcal{O}_x)$, and we define $e_S(g)$ to be $\mathcal{E}_g$.

2.2.2. *Level structures.* One can generalize Bun_G to G-torsors with level structures. Fix a finite set $S \subset |X|$, and for each $x \in S$ let $\mathbf{K}_x \subset L_x G$ be a proalgebraic subgroup commensurable with $L^+_x G$. Then we may talk about G-torsors over X with $\mathbf{K}_S$-level structures: these are G-torsors $\mathcal{E}$ over X together with trivializations $\iota_x : \mathcal{E}|_{\mathrm{Spec}\mathcal{O}_x} \cong G|_{\mathrm{Spec}\mathcal{O}_x}$ (for each $x \in S$) up to left multiplication by $\mathbf{K}_x$ (via the intuitive action if $\mathbf{K}_x \subset L^+_x G$, but it requires some thought to define the action in general). We shall denote the corresponding moduli stack by $\mathrm{Bun}_G(\mathbf{K}_S)$. Then the isomorphism (2.1) generalizes to an isomorphism of groupoids

$$G(F)\backslash G(\mathbb{A}_F)/\prod_{x\notin S}G(\mathcal{O}_x)\times\prod_{x\in S}\mathbf{K}_x(k_x)\xrightarrow{\sim}\mathrm{Bun}_G(\mathbf{K}_S)(k). \tag{2.2}$$

2.3. **Sheaf-to-function correspondence.**

2.3.1. *The dictionary.* Let X be a scheme of finite type over a finite field k and let $\mathcal{F}$ be a constructible complex of $\overline{\mathbb{Q}}_\ell$-sheaves for the étale topology of X. For each closed point $x\in X$, the geometric Frobenius element Frob_x at x acts on the geometric stalk $\mathcal{F}_{\overline{x}}$, which is a complex of $\overline{\mathbb{Q}}_\ell$-vector spaces. We consider the function

$$\begin{array}{rcl} f_{\mathcal{F},k}:X(k) & \to & \overline{\mathbb{Q}}_\ell \\ x & \mapsto & \displaystyle\sum_{i\in\mathbb{Z}}(-1)^i\mathrm{Tr}(\mathrm{Frob}_x,\mathrm{H}^i\mathcal{F}_{\overline{x}}) \end{array}$$

Similarly we can define a function $f_{\mathcal{F},k'}:X(k')\to\overline{\mathbb{Q}}_\ell$ for any finite extension k' of k. The correspondence

$$\mathcal{F}\mapsto\{f_{\mathcal{F},k'}\}_{k'/k}$$

is called the *sheaf-to-function* correspondence. This construction enjoys various functorial properties. For a morphism $\phi:X\to Y$ over k, the derived push forward $f_!$ transforms into integration of functions along the fibers (this is a consequence of the Lefschetz trace formula for the Frobenius endomorphism); the derived pullback f^* transforms into pullback of functions. It also transforms tensor product of sheaves into pointwise multiplication of functions.

2.3.2. **Definition.** Let L be a connected algebraic group over k with the multiplication map $m:L\times L\to L$ and the identity point $e:\mathrm{Spec}k\to L$. A *rank one character sheaf* $\mathcal{K}$ on L is a local system of rank one on L equipped with two isomorphisms

$$\mu:m^*\mathcal{K}\xrightarrow{\sim}\mathcal{K}\boxtimes\mathcal{K},$$
$$u:\overline{\mathbb{Q}}_\ell\xrightarrow{\sim}e^*\mathcal{K}.$$

These isomorphisms should be compatible in the sense that

$$\mu|_{L\times\{e\}}=\mathrm{id}_{\mathcal{K}}\otimes u:\mathcal{K}=\overline{\mathbb{Q}}_\ell\otimes_{\overline{\mathbb{Q}}_\ell}\mathcal{K}\xrightarrow{\sim}e^*\mathcal{K}\otimes\mathcal{K},$$
$$\mu|_{\{e\}\times L}=u\otimes\mathrm{id}_{\mathcal{K}}:\mathcal{K}=\mathcal{K}\otimes_{\overline{\mathbb{Q}}_\ell}\overline{\mathbb{Q}}_\ell\xrightarrow{\sim}\mathcal{K}\otimes e^*\mathcal{K}.$$

Since L is connected, the isomorphism μ in Definition 2.3.2 automatically satisfies the usual cocycle relation on L^3. A local system $\mathcal{K}$ of rank one on L being a character sheaf is a property rather than extra structure on $\mathcal{K}$. Let $\mathcal{CS}_1(L)$ be the category (groupoid) of rank one character sheaves $(\mathcal{K},\mu,u)$ on L, then it carries a symmetric monoidal structure given by the tensor product of character sheaves with the unit object given by the constant sheaf. Let $\mathrm{CS}_1(L)$ be set of isomorphism classes of objects in $\mathcal{CS}_1(L)$, then it is an abelian group. The groupoid $\mathcal{CS}_1(L)$ has trivial automorphisms, and hence is equivalent to the set $\mathrm{CS}_1(L)$. We can similarly define the notion of rank one character sheaves over $\overline{k}$ and form the category $\mathcal{CS}_1(L/\overline{k})$ and the abelian group $\mathrm{CS}_1(L/\overline{k})$. The base change map $\mathrm{CS}_1(L)\to\mathrm{CS}_1(L/\overline{k})$ is injective, and the image consists of $\mathrm{Gal}(\overline{k}/k)$-invariants.

2.3.3. **Remark.** One can make a similar definition of rank one character sheaves when L is not necessarily connected, then the cocycle condition has to be imposed on μ as an extra requirement and μ itself is an extra datum. For example, if L is a discrete group scheme over $\overline{k}$ and we fix an isomorphism $\mathcal{K} \cong \overline{\mathbb{Q}}_\ell$ (as local systems) extending u, then μ (which satisfies the cocycle relation) gives a cocycle $\xi \in Z^2(L, \overline{\mathbb{Q}}_\ell^\times)$ satisfying $\xi_{1,\gamma} = \xi_{\gamma,1} = 1$ for all $\gamma \in L$.

For the rest of this subsection we assume k is finite. For each $\mathcal{K} \in \mathcal{CS}_1(L)$, the sheaf-to-function correspondence gives a function $f_\mathcal{K} : L(k) \to \overline{\mathbb{Q}}_\ell^\times$ which is in fact a group homomorphism because of the isomorphism μ. This way we obtain a homomorphism

$$f_L : \mathrm{CS}_1(L) \to \mathrm{Hom}(L(k), \overline{\mathbb{Q}}_\ell^\times). \tag{2.3}$$

One can show that f_L is always injective. The following result gives descriptions of $\mathrm{CS}_1(L)$ in various special cases.

2.3.4. **Theorem.** (1) *Let L be a connected commutative algebraic group over k. Then f_L is an isomorphism of abelian groups*

$$f_L : \mathrm{CS}_1(L) \xrightarrow{\sim} \mathrm{Hom}(L(k), \overline{\mathbb{Q}}_\ell^\times).$$

(2) *Let L be a connected reductive group over k and $L^{\mathrm{sc}} \to L$ be the simply-connected cover of its derived group. Then f_L induces an isomorphism of abelian groups*

$$\mathrm{CS}_1(L) \xrightarrow{\sim} \mathrm{Hom}(L(k)/L^{\mathrm{sc}}(k), \overline{\mathbb{Q}}_\ell^\times).$$

Let T be a maximal torus in L and $T^{\mathrm{sc}} \subset L^{\mathrm{sc}}$ be its preimage in L^{sc}. Then we also have

$$\mathrm{CS}_1(L) \xrightarrow{\sim} \mathrm{Hom}(T(k)/T^{\mathrm{sc}}(k), \overline{\mathbb{Q}}_\ell^\times).$$

The construction of an inverse to f_L uses the Lang torsor $L \to L$ sending $g \mapsto F_L(g)g^{-1}$, where F_L is the Frobenius endomorphism of L (relative to k).

2.3.5. **Example.** When $L = T$ is a split torus over $k = \mathbb{F}_q$, any object in $\mathcal{CS}_1(L/\overline{k})$ is obtained as a direct summand of $[n]_!\overline{\mathbb{Q}}_\ell$ where $[n] : T \to T$ is the n-th power morphism and n is prime to $\mathrm{char}(k)$. An object $\mathcal{K} \in \mathcal{CS}_1(L/\overline{k})$ belongs to $\mathcal{CS}_1(L)$ if and only if $\mathcal{K}^{\otimes(q-1)} \cong \overline{\mathbb{Q}}_\ell$, or equivalently it is a direct summand of $[q-1]_!\overline{\mathbb{Q}}_\ell$. Objects in $\mathcal{CS}_1(L)$ are also called *Kummer sheaves*.

2.3.6. **Example.** When $L = \mathbb{G}_a$ is the additive group over $k = \mathbb{F}_q$, let $\psi : k \to \overline{\mathbb{Q}}_\ell^\times$ be an additive character. Let $\lambda_{\mathbb{G}_a} : \mathbb{G}_a \to \mathbb{G}_a$ be the $\mathbb{G}_a(k)$-torsor given by $a \mapsto a^q - a$. Objects in $\mathcal{CS}_1(L)$ are of the form $\mathrm{AS}_\psi := (\lambda_{\mathbb{G}_a,!}\overline{\mathbb{Q}}_\ell)_\psi$, the direct summand of $\lambda_{\mathbb{G}_a,!}\overline{\mathbb{Q}}_\ell$ on which $\mathbb{G}_a(k)$ acts through ψ. Such local systems are called *Artin-Schreier sheaves*. Now fix a nontrivial additive character ψ of k. Let $L = V$ be a vector space over k viewed as an additive group. Then objects in $\mathcal{CS}_1(V)$ are of the form $\mathrm{AS}_\phi := \phi^*\mathrm{AS}_\psi$ for a unique $\phi \in V^*$ viewed as a homomorphism $\phi : V \to \mathbb{G}_a$.

2.3.7. **Remark.** (1) Definition 2.3.2 makes sense for any base field k. In this generality we have an alternative way to describe rank one character sheaves. Let $1 \to C \to \widetilde{L} \xrightarrow{v} L \to 1$ be a central extension of algebraic groups with $\widetilde{L}$ connected and C a *discrete* finite group scheme over k.For any character $\chi_C : C \to \overline{\mathbb{Q}}_\ell^\times$, the corresponding direct summand $(v_!\overline{\mathbb{Q}}_\ell)_{\chi_C} \in \mathcal{CS}_1(L)$. Conversely, all objects in $\mathcal{CS}_1(L)$ arise this way.

(2) One can define the category $\mathcal{CS}_1(L)$ for L proalgebraic and connected. In fact, write L as the inverse limit of finite-dimensional quotients L_i, and define $\mathcal{CS}_1(L)$ as the direct 2-limit of $\mathcal{CS}_1(L_i)$.

2.4. **Geometric automorphic data.** We resume with the setting in §2.1. Let $S \subset |X|$ be finite.

2.4.1. **Definition.** A pair $(\mathbf{K}_S, \mathcal{K}_S)$ is a *geometric automorphic datum* with respect to S if

(1) $\mathbf{K}_S$ is a collection $\{\mathbf{K}_x\}_{x\in S}$, where $\mathbf{K}_x \subset L_xG$ is a connected proalgebraic subgroup which is commensurable with L_x^+G (i.e., $\mathbf{K}_x \cap L_x^+G$ has finite codimension in both $\mathbf{K}_x$ and L_x^+G).
(2) $\mathcal{K}_S$ is a collection $\{\mathcal{K}_x\}_{x\in S}$ where $\mathcal{K}_x \in \mathcal{CS}_1(\mathbf{K}_x)$.

Recall from Remark 2.3.7(2) that $\mathcal{K}_x \in \mathcal{CS}_1(\mathbf{K}_x)$ means that $\mathcal{K}_x$ is the pullback of a rank one character sheaf from a finite-dimensional quotient $\mathbf{K}_x \twoheadrightarrow L_x$.

2.4.2. **Remark.** (1) Since k is a finite field, the character sheaf $\mathcal{K}_x$ is uniquely determined by the associated character $\chi_x : \mathbf{K}_x(k_x) \to \overline{\mathbb{Q}}_\ell^\times$ via the sheaf-to-function correspondence (see (2.3)). Therefore we also call the pair $(\mathbf{K}_S, \chi_S)$ a geometric automorphic datum, provided χ_x does arise from an object in $\mathcal{CS}_1(\mathbf{K}_x)$ for all $x \in S$.

(2) Definition 2.4.1 makes sense over any base field k, and we shall work in such generality in §4.

2.4.3. **Definition.** Let $(\mathbf{K}_S, \chi_S)$ be a geometric automorphic datum. An automorphic representation π of $G(\mathbb{A}_F)$ is called $(\mathbf{K}_S, \chi_S)$-*typical* if

(1) For every $x \in S$, the local component π_x of π has a nonzero vector on which $\mathbf{K}_x(k_x)$ acts via the character χ_x.
(2) For every $x \notin S$, π_x is spherical; i.e., $\pi_x^{G(\mathcal{O}_x)} \neq 0$.

Consider the space of functions

$$C_G(\mathbf{K}_S, \chi_S) := C(G(F)\backslash G(\mathbb{A}_F)/\prod_{x\notin S} G(\mathcal{O}_x) \times \prod_{x\in S}(\mathbf{K}_x(k_x), \chi_x), \overline{\mathbb{Q}}_\ell). \tag{2.4}$$

which are invariant under $\prod_{x\notin S} G(\mathcal{O}_x)$ and are eigenvectors under the action of $\mathbf{K}_x(k_x)$ with eigenvalue χ_x, for all $x \in S$.

2.4.4. **Lemma.** *Let $(\mathbf{K}_S, \chi_S)$ be a geometric automorphic datum. Then*

(1) *A necessary condition for $(\mathbf{K}_S, \chi_S)$-typical automorphic representations to exist is that $C_G(\mathbf{K}_S, \chi_S) \neq 0$.*
(2) *If there is a compact subset $\Sigma \subset G(F)\backslash G(\mathbb{A}_F)$ such that all functions in $C_G(\mathbf{K}_S, \chi_S)$ vanish outside Σ, then any $(\mathbf{K}_S, \chi_S)$-typical automorphic representation is cuspidal. In this case, $C_G(\mathbf{K}_S, \chi_S)$ is the direct sum of $(\otimes_{x\notin S}\pi_x^{G(\mathcal{O}_x)}) \otimes (\otimes_{x\in S}\pi_x^{(\mathbf{K}_x(k_x),\chi_x)})$ over $(\mathbf{K}_S, \chi_S)$-typical automorphic representations π.*

2.4.5. *Base change of geometric automorphic data.* Let k'/k be a finite extension. Let S' be the preimage of S in $X \otimes_k k'$ (S' may have more elements than S). Given a geometric automorphic datum $(\mathbf{K}_S, \chi_S)$, we may define a corresponding geometric automorphic datum $(\mathbf{K}_{S'}, \chi_{S'})$ for G and the function field $F' = F \otimes_k k'$. For each $y \in S'$ with image $x \in S$, let $\mathbf{K}_y = \mathbf{K}_x \otimes_{k_x} k_y$. Suppose χ_x corresponds to the character sheaf $\mathcal{K}_x$ on $\mathbf{K}_x$, χ_y is then the character of $\mathbf{K}_y(k_y)$ corresponding to the pullback of $\mathcal{K}_x$ along $\mathbf{K}_y \to \mathbf{K}_x$.

2.4.6. **Definition.** Let Z be the center of G (note by our assumption Z is finite over k). A central character $\omega : Z(F)\backslash Z(\mathbb{A}_F) \to \overline{\mathbb{Q}}_\ell^\times$ is said to be *compatible with the geometric automorphic datum* $(\mathbf{K}_S, \chi_S)$, if ω is unramified away rom S, and $\omega_x|_{Z(F_x)\cap\mathbf{K}_x(k_x)} = \chi_x|_{Z(F_x)\cap\mathbf{K}_x(k_x)}$ for each $x \in S$.

Clearly, the central character ω of a $(\mathbf{K}_S, \chi_S)$-typical automorphic representation is compatible with $(\mathbf{K}_S, \chi_S)$.

2.4.7. **Definition.** Let $(\mathbf{K}_S, \chi_S)$ be a geometric automorphic datum.

(1) It is called *strongly rigid*, if for *every* finite extension k'/k, there are at most one $(\mathbf{K}_{S'}, \chi_{S'})$-typical automorphic representation of $G(\mathbb{A}_{F'})$ (where $F' = F \otimes_k k'$) with any given central character, and if for *some* finite extension k'/k, a $(\mathbf{K}_{S'}, \chi_{S'})$-typical automorphic representation exists.

(2) It is called *weakly rigid*, if there is a constant N such that for *every* finite extension k'/k, there are at most N $(\mathbf{K}_{S'}, \chi_{S'})$-typical automorphic representations of $G(\mathbb{A}_{F'})$.

2.5. **Automorphic sheaves.** Suggested by Weil's interpretation in §2.2 and the sheaf-to-function correspondence §2.3, we shall seek to upgrade the function space (2.4) into a category of sheaves on the moduli stack of G-torsors over X with level structures. This idea was implemented by Drinfeld in the case $G = \mathrm{GL}_2$, and for general G it has been formulated as the geometric Langlands correspondence by Laumon [24] and Beilinson and Drinfeld [2].

2.5.1. *The category of automorphic sheaves.* Consider the situation in §2.4. Let $\mathbf{K}_x^+ \subset \mathbf{K}_x$ be a connected normal subgroup of finite codimension such that the rank one character sheaf $\mathcal{K}_x$ on $\mathbf{K}_x$ is pulled back from the quotient $L_x = \mathbf{K}_x/\mathbf{K}_x^+$. Let $\mathrm{Bun}_G(\mathbf{K}_S)$ and $\mathrm{Bun}_G(\mathbf{K}_S^+)$ be the moduli stack of G-torsors over X with the respective level structures, as defined in §2.2.2. The morphism $\mathrm{Bun}_G(\mathbf{K}_S^+) \to \mathrm{Bun}_G(\mathbf{K}_S)$ is an $L_S := \prod_{x \in S} L_x$-torsor. The tensor product $\mathcal{K}_S := \boxtimes_{x \in S} \mathcal{K}_x$ is an object in $\mathcal{CS}_1(L_S)$. It makes sense to talk about $\overline{\mathbb{Q}}_\ell$-complexes of sheaves on $\mathrm{Bun}_G(\mathbf{K}_S^+)$ which are $(L_S, \mathcal{K}_S)$-equivariant.

Without giving the detailed definition, we have the derived category

$$D_G(\mathbf{K}_S, \chi_S) := D^b_{(L_S, \mathcal{K}_S)}(\mathrm{Bun}_G(\mathbf{K}_S^+), \overline{\mathbb{Q}}_\ell)$$

consisting of bounded constructible $\overline{\mathbb{Q}}_\ell$-complexes on $\mathrm{Bun}_G(\mathbf{K}_S^+)$ equipped with $(L_S, \mathcal{K}_S)$-equivariant structures. There is subtle in defining this category because $\mathrm{Bun}_G(\mathbf{K}_S^+)$ is a stack that is not of finite type. However we will only consider sheaves that are supported on finite-type substacks of $\mathrm{Bun}_G(\mathbf{K}_S^+)$. Objects in the category $D_G(\mathbf{K}_S, \chi_S)$ are called *automorphic sheaves* with respect to the geometric automorphic datum $(\mathbf{K}_S, \chi_S)$.

2.5.2. *Relevant points.* Consider a point $\mathcal{E} \in \mathrm{Bun}_G(\mathbf{K}_S)(\overline{k})$, which represents a G-torsor over $X_{\overline{k}}$ with $\mathbf{K}_y$-level structures at $y \in S(\overline{k})$. The automorphism group $\mathrm{Aut}(\mathcal{E})$ of the point $\mathcal{E}$ is an affine algebraic group over $\overline{k}$. For each $y \in S(\overline{k})$ with image $x \in S$, restricting an automorphism of $\mathcal{E}$ gives an element in $\mathbf{K}_y = \mathbf{K}_x \otimes_{k_x} \overline{k}$ (this depends on the choice a trivialization of $\mathcal{E}$ around x), and thus a homomorphism

$$\mathrm{ev}_{S,\mathcal{E}} : \mathrm{Aut}(\mathcal{E}) \to \prod_{y \in S(\overline{k})} \mathbf{K}_y \to \prod_{y \in S(\overline{k})} L_y = L_S \otimes_k \overline{k} \tag{2.5}$$

which is well-defined up to conjugacy if one changes the trivialization of $\mathcal{E}$ around x.

2.5.3. **Definition.** Let $(\mathbf{K}_S, \chi_S)$ be a geometric automorphic datum. A point $\mathcal{E} \in \mathrm{Bun}_G(\mathbf{K}_S)(\overline{k})$ is *relevant to* $(\mathbf{K}_S, \chi_S)$ if the restriction of $\mathrm{ev}^*_{S,\mathcal{E}}\mathcal{K}_S$ to the neutral component $\mathrm{Aut}^\circ(\mathcal{E})$ of $\mathrm{Aut}(\mathcal{E})$ is isomorphic to the constant sheaf. Otherwise the point $\mathcal{E}$ is called *irrelevant*.

According to (2.2), we may identify the double coset $G(F)\backslash G(\mathbb{A}_F)/\prod_{x \notin S} G(\mathcal{O}_x) \times \prod_{x \in S} \mathbf{K}_x^+$ with $\mathrm{Bun}_G(\mathbf{K}_S^+)(k)$, and view functions in $C_G(\mathbf{K}_S, \chi_S)$ as functions on the underlying set of $\mathrm{Bun}_G(\mathbf{K}_S^+)(k)$ that are (L_S, χ_S)-equivariant. The sheaf-to-function correspondence then gives an additive map

$$\mathrm{Ob}\ D_G(\mathbf{K}_S, \chi_S) \to C_G(\mathbf{K}_S, \chi_S).$$

We shall also need to consider similar objects after a base change from k to a finite extension k' or to $k' = \overline{k}$. We shall use the notation

$$D_G(k';\mathbf{K}_S, \chi_S) \text{ and } C_G(k';\mathbf{K}_S, \chi_S)$$

to denote the analogs of $D_G(\mathbf{K}_S, \chi_S)$ and $C_G(\mathbf{K}_S, \chi_S)$ defined for the function field $F' = F \otimes_k k'$ and the automorphic data $(\mathbf{K}_{S'}, \chi_{S'})$ obtained by base change from $(\mathbf{K}_S, \chi_S)$ as in §2.4.5.

2.5.4. **Lemma.** (1) *Let $\mathcal{E} \in \mathrm{Bun}_G(\mathbf{K}_S)(\overline{k})$ be an irrelevant point. Then for any object $\mathcal{F} \in D_G(\overline{k};\mathbf{K}_S, \chi_S)$, $i^*_{\mathcal{E}}\mathcal{F} = 0$ and $i^!_{\mathcal{E}}\mathcal{F} = 0$. Here $i_{\mathcal{E}}$ denotes the inclusion map of the fiber of $\mathcal{E}$ in $\mathrm{Bun}_G(\mathbf{K}^+_S)$.*

(2) *Let $[g] \in G(F)\backslash G(\mathbb{A}_F)/\prod_{x\notin S} G(\mathcal{O}_x) \times \prod_{x\in S} \mathbf{K}_x(k_x) = \mathrm{Bun}_G(\mathbf{K}_S)(k)$ be an irrelevant point (when viewed as a $\overline{k}$-point). Then any $f \in C_G(\mathbf{K}_S, \chi_S)$ vanishes on the preimage of $[g]$ in $G(F)\backslash G(\mathbb{A}_F)/\prod_{x\notin S} G(\mathcal{O}_x) \times \prod_{x\in S} \mathbf{K}^+_x(k_x)$. Similar statement holds when k is replaced by a finite extension k'.*

2.5.5. **Lemma.** *Let $(\mathbf{K}_S, \chi_S)$ be a geometric automorphic datum.*

(1) *If there is no relevant $\overline{k}$-point on $\mathrm{Bun}_G(\mathbf{K}_S)$, then $D_G(k';\mathbf{K}_S, \chi_S) = 0$ and $C_G(k';\mathbf{K}_S, \chi_S) = 0$ for any finite extension k' of k.*

(2) *If $\mathrm{Bun}_G(\mathbf{K}_S)$ contains only finitely many relevant $\overline{k}$-points, then $(\mathbf{K}_S, \chi_S)$ is weakly rigid. Moreover for any finite extension k' of k, any $(\mathbf{K}'_S, \chi'_S)$-typical automorphic representation of $G(\mathbb{A}_{F'})$ is cuspidal.*

2.5.6. **Example.** Let $G = \mathrm{SL}_2$ and $F = k(t)$, the function field of $X = \mathbb{P}^1_k$. Let $S = \{0, 1, \infty\} \subset |\mathbb{P}^1_k| = |X|$. For each closed point $x \in |X|$, we let $\mathbf{K}_x = \mathbf{I}_x$ be the standard Iwahori subgroup of L_xG, i.e.,

$$\mathbf{I}_x(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_x) | c \in \mathfrak{m}_x \right\}.$$

For each $x \in S$, we choose a character $\chi_x : k^\times \to \overline{\mathbb{Q}}^\times_\ell$ and view it as a character of $\mathbf{I}_x(k)$ by sending $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\chi_x(\overline{a})$, where $\overline{a} \in k^\times$ is the image of $a \in \mathcal{O}^\times_x$. We consider the geometric automorphic datum $(\mathbf{I}_S, \chi_S)$.

A central character ω compatible with $(\mathbf{I}_S, \chi_S)$ exists if and only if

$$\prod_{x\in S} \chi_x(-1) = 1. \tag{2.6}$$

If this condition is satisfied, the central character ω compatible with $(\mathbf{I}_S, \chi_S)$ is unique. The condition (2.6) can always be satisfied by passing to a quadratic extension k'/k and using the base-changed automorphic datum $(\mathbf{I}_{S'}, \chi_{S'})$.

2.5.7. **Proposition.** *Suppose for any map $\epsilon : S \to \{\pm 1\}$, $\prod_{x\in S} \chi_x^{\epsilon(x)}$ is not the trivial character on $k^\times$. Then the geometric automorphic datum $(\mathbf{I}_S, \chi_S)$ is strictly rigid.*

The moduli stack $\mathrm{Bun}_G(\mathbf{I}_S)$ in this case classifies $(\mathcal{V}, \iota, \{\ell_x\}_{x\in S})$ where $\mathcal{V}$ is a rank two vector bundle over X, $\iota : \wedge^2(\mathcal{V}) \cong \mathcal{O}_X$ and ℓ_x is a line in the fiber $\mathcal{V}_x$. There is a unique relevant point in $\mathrm{Bun}_G(\mathbf{I}_S)$, which corresponds to the trivial bundle $\mathcal{O}^2_X$ with three lines $\{\ell_x\}_{x\in S}$ in generic position (i.e., three distinct lines in k^2).

3. Rigidity for local systems

In this section k is any field. We shall review the notion of a local system in étale topology, and introduce the notion of rigidity for them.

3.1. **Local systems.**

3.1.1. *Local systems in étale topology.* Let U be a scheme of finite type over k. We recall some definitions from [18, §1.2, §1.4.2, §1.4.3]. A $\mathbb{Z}_\ell$-local system on U is a projective system $(\mathcal{F}_n)_{n\geq 1}$ of locally constant locally free $\mathbb{Z}/\ell^n\mathbb{Z}$-sheaves $\mathcal{F}_n$ of finite rank on X under the étale topology such that the natural map $\mathcal{F}_n \otimes \mathbb{Z}/\ell^{n-1}\mathbb{Z} \to \mathcal{F}_{n-1}$ is an isomorphism for all n. Denote the category of $\mathbb{Z}_\ell$-local systems on U by $\mathrm{Loc}(U, \mathbb{Z}_\ell)$, which is a $\mathbb{Z}_\ell$-linear abelian category. The category of $\mathbb{Q}_\ell$-local systems is by definition the abelian category $\mathrm{Loc}(U, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell$ obtained by inverting ℓ in the Hom groups in $\mathrm{Loc}(U, \mathbb{Z}_\ell)$. Similar definition gives $\mathrm{Loc}(U, \mathcal{O}_E)$ and $\mathrm{Loc}(U, E)$ for any finite extension E of $\mathbb{Q}_\ell$. Finally define $\mathrm{Loc}(U) := \mathrm{Loc}(U, \overline{\mathbb{Q}}_\ell)$ to be the inductive limit $\varinjlim_E \mathrm{Loc}(U, E)$ over all finite extensions E of $\mathbb{Q}_\ell$.

In the sequel we assume U is connected. Fix a geometric point $u \in U$. Let $\mathcal{F}$ be an $\overline{\mathbb{Q}}_\ell$-local system on U of rank n. The stalk $\mathcal{F}_u$ is an $\overline{\mathbb{Q}}_\ell$-vector space of dimension n which carries the action of the étale fundamental group $\pi_1(U, u)$ defined in [12, V,§7]. Thus $\mathcal{F}$ determines a *continuous* homomorphism

$$\pi_1(U, u) \to \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(\mathcal{F}_u) \cong \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell),$$

where $\overline{\mathbb{Q}}_\ell$ is topologized as the inductive limit of finite extensions E of $\mathbb{Q}_\ell$, each with the ℓ-adic topology. The last isomorphism above depends upon a choice of a basis of $\mathcal{F}_u$. We get a functor

$$\omega_u : \mathrm{Loc}(U) \to \mathrm{Rep}_{\mathrm{cont}}(\pi_1(U, u), \overline{\mathbb{Q}}_\ell) \tag{3.1}$$

where $\mathrm{Rep}_{\mathrm{cont}}(\pi_1(U, u), \overline{\mathbb{Q}}_\ell)$ is the category of continuous representation of $\pi_1(U, u)$ on finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces. Both sides of (3.1) carry tensor structures and ω_u is in fact an equivalence of tensor categories ([18, Proposition 1.2.5]).

Suppose furthermore that U is normal and connected and $F = k(U)$ is the function field of U. When u is a geometric generic point of X, $\pi_1(U, u)$ is a quotient of the absolute Galois group $\mathrm{Gal}(F^s/F)$ ([12, V, Proposition 8.2]). Therefore every local system of rank n over U determines a continuous Galois representations

$$\rho : \mathrm{Gal}(F^s/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell).$$

We shall denote by $\pi_1^{\mathrm{geom}}(U, u)$ the fundamental group of $\pi_1(U \otimes_k \overline{k}, u)$, and call it the *geometric fundamental group* of U (with respect to the base point u). This is a normal subgroup of $\pi_1(U, u)$ which fits into an exact sequence ([12, V, Proposition 6.13])

$$1 \to \pi_1^{\mathrm{geom}}(U, u) \to \pi_1(U, u) \to \mathrm{Gal}(k^s/k) \to 1.$$

3.1.2. *H-local systems.* Let H be an affine algebraic group over $\overline{\mathbb{Q}}_\ell$ of finite type. We may also define the notion of H-local system on a connected U. There are two ways to do this.

First definition. Fix a geometric point $u \in U$. An H-local system on U is a continuous homomorphism

$$\rho : \pi_1(U, u) \to H(\overline{\mathbb{Q}}_\ell). \tag{3.2}$$

Such homomorphisms form a category $\mathrm{Loc}_H(U)$, in which isomorphisms are given by $H(\overline{\mathbb{Q}}_\ell)$-conjugacy of representations. For example, $\mathrm{Loc}_{\mathrm{GL}_n}(U)$ is the full subcategory of $\mathrm{Loc}(U)$ consisting of local systems of rank n.

Second (and more canonical) definition. Let $\mathrm{Rep}(H, \overline{\mathbb{Q}}_\ell)$ be the tensor category of algebraic representations of H on finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces. We define the category

$$\mathrm{Loc}_H(U) := \mathrm{Fun}^{\otimes}(\mathrm{Rep}(H, \overline{\mathbb{Q}}_\ell), \mathrm{Loc}(U)).$$

Here $\mathrm{Fun}^{\otimes}$ denotes the category of tensor functors between two tensor categories.

The two notions of H-local systems are equivalent. Given a representation ρ as in (3.2) and for $V \in \mathrm{Rep}(H, \overline{\mathbb{Q}}_\ell)$, the composition

$$\rho_V : \pi_1(U, u) \xrightarrow{\rho} H(\overline{\mathbb{Q}}_\ell) \to \mathrm{GL}(V)$$

is an object in $\mathrm{Loc}(U)$ of rank equal to $\dim V$. The assignment $V \mapsto \rho_V$ gives a tensor functor $\mathrm{Rep}(H, \overline{\mathbb{Q}}_\ell) \to \mathrm{Loc}(U)$. Conversely, given a tensor functor $\mathcal{F} : \mathrm{Rep}(H, \overline{\mathbb{Q}}_\ell) \to \mathrm{Loc}(U)$, using the equivalence (3.1), this can be viewed as a tensor functor $\mathrm{Rep}(H, \overline{\mathbb{Q}}_\ell) \to \mathrm{Rep}_{\mathrm{cont}}(\pi_1(U, u), \overline{\mathbb{Q}}_\ell)$. The Tannakian formalism [7] then implies that such a tensor functor comes from a group homomorphism ρ as in (3.2), well-defined up to conjugacy.

3.1.3. **Definition.** Let $\rho : \pi_1(U, u) \to H(\overline{\mathbb{Q}}_\ell)$ be an H-local system. The *global geometric monodromy group* H^{geom}_ρ of the H-local system ρ is the Zariski closure of $\rho(\pi_1^{\mathrm{geom}}(U, u))$ in H.

3.1.4. *Local monodromy.* Let X be a projective, smooth and geometrically connected curve over a perfect field k. We fix a finite set of closed points $S \subset |X|$ and let $U = X - S$. Let $j : U \hookrightarrow X$ be the open inclusion. Let $x \in S$, and let $i_x : \{x\} = \mathrm{Spec} k_x \hookrightarrow X$ be the inclusion. Fix an algebraic closure $\overline{F}_x$ of F_x. This gives a geometric generic point $\eta_x \in X$. The morphism $\mathrm{Spec} F_x \to U$ then induces an injective homomorphism of fundamental groups

$$\mathrm{Gal}(F_x^s/F_x) \hookrightarrow \pi_1(U, \eta_x) \cong \pi_1(U, u), \tag{3.3}$$

where the second map is well-defined up to conjugacy. Since F_x is a complete discrete valuation field with perfect residue field k_x, we have an exact sequence

$$1 \to \mathcal{I}_x \to \mathrm{Gal}(F_x^s/F_x) \to \mathrm{Gal}(\overline{k}/k_x) \to 1.$$

The normal subgroup $\mathcal{I}_x$ of $\mathrm{Gal}(F_x^s/F_x)$ is the *inertia group* at x. Under (3.3), $\mathcal{I}_x$ is contained in the normal subgroup $\pi_1^{\mathrm{geom}}(U, u) \lhd \pi_1(U, u)$. By the *local monodromy* of ρ at $x \in S$ we mean the restriction $\rho|_{\mathcal{I}_x}$.

When $\mathrm{char}(k) = p > 0$, We have a normal subgroup $\mathcal{I}_x^w \lhd \mathcal{I}_x$ called the *wild inertia group* such that the quotient $\mathcal{I}_x^t := \mathcal{I}_x/\mathcal{I}_x^w$ is the maximal prime-to-p quotient of $\mathcal{I}_x$, called the *tame inertia group*. We have a canonical isomorphism of $\mathrm{Gal}(\overline{k}/k_x)$-modules

$$\mathcal{I}_x^t \xrightarrow{\sim} \varprojlim_{(n,p)=1} \mu_n(\overline{k}).$$

The local system ρ is said to be *tame at* $x \in S$ if $\rho|_{\mathcal{I}_x}$ factors through the tame inertia group $\mathcal{I}_x^t$.

3.2. **Two notions of rigidity for local systems.** Rigidity of a local system is a geometric property, therefore we assume the base field k to be algebraically closed in this subsection. Let X be a complete smooth connected algebraic curve over k. Fix an open subset $U \subset X$ with finite complement S.

3.2.1. **Definition** (extending Katz [20, §1.0.3])**.** Let H be an algebraic group over $\overline{\mathbb{Q}}_\ell$. Let ρ be an H-local system on U. Then ρ is called *physically rigid* if, for any other H-local system ρ' such that $\rho'|_{\mathcal{I}_x} \cong \rho|_{\mathcal{I}_x}$ (meaning conjugate in $H(\overline{\mathbb{Q}}_\ell)$) for each $x \in S$, we have $\rho \cong \rho'$ as objects in $\mathrm{Loc}_H(U)$.

Although the definition uses U as an input, the notion of physical rigidity is in fact independent of the open subset U: for any nonempty open subset $V \subset U$, ρ is rigid over U if and only if $\rho|_V$ is rigid over V. Therefore, physical rigidity is a property of the Galois representation $\rho_\eta : \mathrm{Gal}(F^s/F) \to H(\overline{\mathbb{Q}}_\ell)$ obtained by restricting ρ to a geometric generic point η of the X.

Next we introduce cohomological rigidity. For this we assume H is a connected semisimple group over $\overline{\mathbb{Q}}_\ell$. Let $\mathfrak{h}$ be the Lie algebra of H, and let $\mathrm{Ad}(\rho)$ be the composition $\pi_1(U) \xrightarrow{\rho}$

$H(\overline{\mathbb{Q}}_\ell) \to \mathrm{GL}(\mathfrak{h})$, viewed as a local system on U of rank $\dim \mathfrak{h}$. Let $j : U \hookrightarrow X$ be the open embedding and $j_{!*}\mathrm{Ad}(\rho)$ be the non-derived direct image of $\mathrm{Ad}(\rho)$ along j. Concretely, the stalk of $j_{!*}\mathrm{Ad}(\rho)$ at $x \in S$ is the $\mathcal{I}_x$-invariants on $\mathfrak{h}$.

3.2.2. **Definition** (extending Katz [20, §5.0.1])**.** Let H be a connected semisimple group over $\overline{\mathbb{Q}}_\ell$. An object $\rho \in \mathrm{Loc}_H(U)$ is called *cohomogically rigid*, if

$$\mathrm{Rig}(\rho) := \mathrm{H}^1(X, j_{!*}\mathrm{Ad}(\rho)) = 0.$$

The vector space $\mathrm{Rig}(\rho)$ does not change if we shrink U to a smaller open subset. Therefore cohomological rigidity is also a property of the Galois representation $\rho_\eta : \mathrm{Gal}(F^s/F) \to H(\overline{\mathbb{Q}}_\ell)$.

3.2.3. **Remark.** When we work over the base field $\mathbb{C}$ and view U as a topological surface, one can define a moduli stack $\mathcal{M}$ of H-local systems over U with prescribed local monodromy around the punctures S. The same formula $\mathrm{Rig}(\rho)$ then calculates the dimension of the tangent space $T_\rho\mathcal{M}$ at ρ. The condition $\mathrm{Rig}(\rho) = 0$ in this topological setting says that ρ does not admit infinitesimal deformations with prescribed local monodromy around S, i.e., $\mathcal{F}$ is rigid. This interpretation is the motivation for Definition 3.2.2. However, defining a moduli stack of ℓ-adic local systems is much subtler, and this topological interpretation only serves as a heuristic.

Using the Grothendieck-Ogg-Shafarevich formula, it is easy to give the following numerical criterion for cohomological rigidity.

3.2.4. **Proposition.** *Let ρ be an H-local system on $U = X - S$. Let g_X be the genus of X. Then ρ is cohomologically rigid if and only if*

$$\frac{1}{2}\sum_{x\in S} a_x(\mathrm{Ad}(\rho)) = \dim \mathfrak{h}/(\mathfrak{h})^{\pi_1(U,u)} - g_X \dim \mathfrak{h}. \tag{3.4}$$

Here $a_x(\mathrm{Ad}(\rho))$ is the Artin conductor of the action of $\mathcal{I}_x$ on $\mathfrak{h}$.

From (3.4) we see that cohomologically rigid H-local systems exist only when $g_X \leq 1$. When $g_X = 1$, rigid examples are very limited (see [20, §1.4]). Most examples of rigid local systems are over open subsets of $\mathbb{P}^1$.

When $H = \mathrm{SL}_n$, the two notions of rigidity are related by the following theorem.

3.2.5. **Theorem** (Katz [20, Theorem 5.0.2])**.** *For $X = \mathbb{P}^1$ and $H = \mathrm{SL}_n$, cohomological rigidity of an SL_n-local system implies its physical rigidity.*

3.2.6. **Remark.** An alternative approach to define the notion of rigidity for a local system ρ over U over a finite field k is by requiring that the adjoint L-function of ρ to be trivial (constant function 1). This is the approach taken by Gross in [14]. When $\mathrm{H}^0(U_{\overline{k}}, \mathrm{Ad}(\rho)) = 0$, triviality of the adjoint L-function of ρ is equivalent to cohomological rigidity of ρ.

3.3. **Rigidity in the inverse Galois theory.** It is instructive to compare the notion of rigidity for local systems with the notion of a rigid tuple in inverse Galois theory. We give a quick review following [33, Chapter 8]. Let H be a *finite* group with trivial center.

3.3.1. **Definition.** A tuple of conjugacy classes $(C_1, C_2, \cdots, C_n)$ in H is called *(strictly) rigid*, if

- The equation
$$g_1 g_2 \cdots g_n = 1 \tag{3.5}$$
has a solution with $g_i \in C_i$, and the solution is unique up to simultaneous H-conjugacy;
- For any solutions $(g_1, \cdots, g_n)$ of (3.5), $\{g_i\}_{i=1,\cdots,n}$ generate H.

The connection between rigid tuples and local systems is given by the following theorem. Let $S = \{P_1, \cdots, P_n\} \subset \mathbb{P}^1(\mathbb{Q})$, and let $U = \mathbb{P}^1_{\mathbb{Q}} - S$.

3.3.2. **Theorem** (Belyi, Fried, Matzat, Shih, and Thompson)**.** *Let $(C_1, \cdots, C_n)$ be a rigid tuple in H. Then up to isomorphism there is a unique connected unramified Galois H-cover $\pi : Y \to U \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ such that a topological generator of the (tame) inertia group at P_i acts on Y as an element in C_i.*

Furthermore, if each C_i is rational (i.e., C_i takes rational values for all irreducible characters of H), then the H-cover $Y \to U \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is defined over $\mathbb{Q}$.

From the above theorem we see that the notion of a rigid tuple is an analog of physical rigidity for H-local systems when the algebraic group H is a finite group.

Rigid tuples combined with the Hilbert irreducibility theorem solves the inverse Galois problem for H.

3.3.3. **Corollary.** *Suppose there exists a rational rigid tuple in H, then H can be realized as* $\mathrm{Gal}(K/\mathbb{Q})$ *for some Galois number field $K/\mathbb{Q}$.*

For a comprehensive survey of finite simple groups that are realized as Galois groups over $\mathbb{Q}$ using rigidity tuples, we refer the readers to the book [27] by Malle and Matzat. Recent work of Guralnick and Malle [16] establishes a rational rigid triple in $E_8(\mathbb{F}_\ell)$ following the suggestion of [36].

4. Calculus of geometric Hecke operators

This is the most technical part of this report. We will review the basic setting of the geometric Langlands program. The main result is Theorem 4.4.3, which roughly says that for rigid automorphic representations, their Galois representations under the Langlands correspondence can be explicitly constructed.

4.1. **Working with a more general base field k.** In this section, we work with geometric objects such as Bun_G and sheaves on them instead of talking about functions on $G(F)\backslash G(\mathbb{A}_F)$. Therefore k can be a more general field. In fact we assume k **is a finite extension of a prime field**, i.e., it is either a finite field or a number field. This assumption is not essential, and is to get the Satake equivalence (see §4.2) to work in the simplest way.

We continue with the situation in §2.1, with all geometric objects, e.g., the curve X and the group G, defined over the more general field k.

By Remark 2.3.7(1), the notion of rank one character sheaves makes sense over k, so is the notion of a geometric automorphic datum in Definition 2.4.1. For general k, we shall write $(\mathbf{K}_S, \mathcal{K}_S)$ for a geometric automorphic datum instead of $(\mathbf{K}_S, \chi_S)$ since the correspondence $\mathcal{K}_x \leftrightarrow \chi_x$ only works for finite fields. As a result, we shall also change a few notations. For example, $D_G(\mathbf{K}_S, \mathcal{K}_S)$ shall replace $D_G(\mathbf{K}_S, \chi_S)$.

We will also be working with derived categories of $\overline{\mathbb{Q}}_\ell$-sheaves and perverse sheaves on algebraic stacks. For foundational material we refer to the articles of Laszlo and Olsson [23] and Y.Liu and W.Zheng [25]. **Note: All sheaf-theoretic functors are derived**.

4.2. **The Satake category.** The Satake category is an upgraded version of the spherical Hecke algebra $C_c(G(\mathcal{O}_x)\backslash G(F_x)/G(\mathcal{O}_x))$ under the sheaf-to-function correspondence.

Let LG and L^+G be the group objects over k defined similarly as L_xG and L^+_xG, using the "standard" local field $k((t))$ in place of F_x. The fppf quotient $\mathrm{Gr} = LG/L^+G$ is called the *affine Grassmannian* of G. Then L^+G acts on Gr via left translation. The L^+G-orbits on Gr are indexed by dominant coweights $\lambda \in \mathbb{X}_*(T)^+$. The orbit containing the element $t^\lambda \in T(k((t)))$ is denoted by Gr_λ and its closure is denoted by $\mathrm{Gr}_{\leq\lambda}$. We have $\dim \mathrm{Gr}_\lambda = \langle 2\rho, \lambda\rangle$, where 2ρ is the sum of positive roots in G. Each $\mathrm{Gr}_{\leq\lambda}$ is a projective but usually singular variety over k. We denote the intersection complex of $\mathrm{Gr}_{\leq\lambda}$ by IC_λ: this is the middle extension of the shifted and twisted constant sheaf $\overline{\mathbb{Q}}_\ell[\langle 2\rho,\lambda\rangle](\langle\rho,\lambda\rangle)$ on Gr_λ (note that $\langle\rho,\lambda\rangle \in \mathbb{Z}$ since G is simply-connected).

Let $\widetilde{\mathrm{Sat}} = \mathrm{Perv}_{L^+G}(\mathrm{Gr})$ be the category of L^+G-equivariant perverse sheaves on Gr (with $\overline{\mathbb{Q}}_\ell$-coefficients) with finite type supports. The *Satake category* $\mathrm{Sat} \subset \widetilde{\mathrm{Sat}}$ is the full subcategory consisting of direct sums of IC_λ's ($\lambda \in \mathbb{X}_*(T)^+$). In [26], [11] and [28], it was shown that when k is algebraically closed, Sat carries a natural tensor structure, such that the global cohomology functor $h = H^*(\mathrm{Gr}, -) : \mathrm{Sat} \to \mathrm{Vec}$ is a fiber functor. It is also shown that the Tannakian group of the tensor category Sat is the Langlands dual group $\widehat{G}$. When k is a finite field or a number field, these results still hold, see the discussion in [36, §4.1.1, towards the end]. The Tannakian formalism gives the *geometric Satake equivalence* of tensor categories

$$\mathrm{Sat} \cong \mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell).$$

For $V \in \mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell)$, we denote the corresponding object in Sat by IC_V.

4.3. **Geometric Hecke operators.** We consider the situation of §2.5.1. In particular, we have a geometric automorphic datum $(\mathbf{K}_S, \mathcal{K}_S)$, and moduli stacks $\mathrm{Bun}_G(\mathbf{K}_S)$ and $\mathrm{Bun}_G(\mathbf{K}_S^+)$.

4.3.1. *Hecke correspondence.* Consider the following diagram

$$\begin{tikzcd} & \mathrm{Hk}(\mathbf{K}_S^+) \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] \arrow[rr, "\pi"] & & U := X - S \\ \mathrm{Bun}_G(\mathbf{K}_S^+) & & \mathrm{Bun}_G(\mathbf{K}_S^+) & \end{tikzcd} \tag{4.1}$$

Here, the stack $\mathrm{Hk}(\mathbf{K}_S^+)$ classifies the data $(x, \mathcal{E}, \mathcal{E}', \tau)$ where $x \in U := X - S$, $\mathcal{E}, \mathcal{E}' \in \mathrm{Bun}_G(\mathbf{K}_S^+)$ and $\tau : \mathcal{E}|_{X-\{x\}} \xrightarrow{\sim} \mathcal{E}'|_{X-\{x\}}$ is an isomorphism of G-torsors over $X - \{x\}$ preserving the $\mathbf{K}_x^+$-level structures at each $x \in S$. The morphisms $\overleftarrow{h}$, $\overrightarrow{h}$ and π send $(x, \mathcal{E}, \mathcal{E}', \tau)$ to $\mathcal{E}, \mathcal{E}'$ and x respectively.

For $x \in U$, we denote its preimage under π by $\mathrm{Hk}_x(\mathbf{K}_S^+)$. We have an evaluation morphism

$$\mathrm{ev}_x : \mathrm{Hk}_x(\mathbf{K}_S^+) \to L_x^+G\backslash L_xG/L_x^+G. \tag{4.2}$$

In fact, for a point $(x, \mathcal{E}, \mathcal{E}', \tau) \in \mathrm{Hk}_x(\mathbf{K}_S^+)$, if we fix trivializations of $\mathcal{E}$ and $\mathcal{E}'$ over $\mathrm{Spec}\mathcal{O}_x$, the isomorphism τ restricted to $\mathrm{Spec}F_x$ is an isomorphism between the trivial G-torsors over $\mathrm{Spec}F_x$, hence given by a point $g_\tau \in L_xG$. Changing the trivializations of $\mathcal{E}|_{\mathrm{Spec}\mathcal{O}_x}$ and $\mathcal{E}'|_{\mathrm{Spec}\mathcal{O}_x}$ will result in left and right multiplication of g_τ by elements in L_x^+G. Therefore we have a well-defined morphism ev_x as above between stacks.

As x moves along U, we may identify the target of (4.2) as $L^+G\backslash LG/L^+G$ by choosing a local coordinate t at x. Modulo the ambiguity caused by the choice of the local coordinates, we obtain a well defined morphism

$$\mathrm{ev} : \mathrm{Hk}(\mathbf{K}_S^+) \to \left[\frac{L^+G\backslash LG/L^+G}{\mathrm{Aut}^+}\right], \tag{4.3}$$

where Aut^+ is the group scheme over k of continuous ring automorphisms of $k[[t]]$, and it acts on LG and L^+G via its action on $k[[t]]$.

4.3.2. *Geometric Hecke operators.* For each object $V \in \mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell)$, the corresponding object $\mathrm{IC}_V \in \mathrm{Sat}$ under the geometric Satake equivalence defines a complex on the quotient stack $\left[\frac{L^+G\backslash LG/L^+G}{\mathrm{Aut}^+}\right]$. The geometric Hecke operator associated with V is the functor

$$\begin{aligned} \mathbb{T}_V : D_{(L_S,\mathcal{K}_S)}(\mathrm{Bun}_G(\mathbf{K}_S^+) \times U) &\to D_{(L_S,\mathcal{K}_S)}(\mathrm{Bun}_G(\mathbf{K}_S^+) \times U) \\ \mathcal{F} &\mapsto (\overrightarrow{h} \times \pi)_! \left((\overleftarrow{h} \times \pi)^* \mathcal{F} \otimes \mathrm{ev}^* \mathrm{IC}_V \right). \end{aligned}$$

The composition of these functors are compatible with the tensor structure of Sat: there is a natural isomorphism of functors

$$\mathbb{T}_V \circ \mathbb{T}_W \cong \mathbb{T}_{V \otimes W}, \forall V, W \in \mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell)$$

which is compatible with the associativity of the tensor product in $\mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell)$ and the associativity of composition of functors $\mathbb{T}_{V_1} \circ \mathbb{T}_{V_2} \circ \mathbb{T}_{V_3}$ in the obvious sense.

4.3.3. **Definition.** A Hecke eigensheaf in $D_G(\mathbf{K}_S, \mathcal{K}_S)$ consists of $\mathcal{F} \in D_G(\mathbf{K}_S, \mathcal{K}_S)$, $\rho \in \mathrm{Loc}_{\widehat{G}}(U)$ and for every $V \in \mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell)$, an isomorphism

$$\varphi_V : \mathbb{T}_V(\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell) \cong \mathcal{F} \boxtimes \rho_V$$

such that the following conditions are satisfied.

(1) As V varies in $\mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell)$, φ_V gives a natural isomorphism between the functors $\mathbb{T}_{(-)}(\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell)$ and $\mathcal{F} \boxtimes \rho_{(-)}$.
(2) For $V = \overline{\mathbb{Q}}_\ell$, we require that φ_V be the identity map for $\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell$.
(3) For $V, W \in \mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell)$, $\varphi_{V \otimes W}$ is equal to the composition

$$\begin{aligned} & \mathbb{T}_{V \otimes W}(\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell) \cong \mathbb{T}_V \circ \mathbb{T}_W(\mathcal{F} \boxtimes \overline{\mathbb{Q}}_\ell) \xrightarrow{\varphi_W} \mathbb{T}_V(\mathcal{F} \boxtimes \rho_W) \\ \cong \quad & \mathbb{T}_V(\mathcal{F} \otimes \overline{\mathbb{Q}}_\ell) \otimes \rho_W \xrightarrow{\varphi_V} \mathcal{F} \boxtimes (\rho_V \otimes \rho_W) \cong \mathcal{F} \boxtimes \rho_{V \otimes W} \end{aligned}$$

We sometime abuse the language and say that $\mathcal{F} \in D_G(\mathbf{K}_S, \mathcal{K}_S)$ is a Hecke eigensheaf, if there exists a $\widehat{G}$-local system ρ and isomorphisms φ_V as in the above definition. In this case, we call ρ the Hecke eigen $\widehat{G}$-local system associated with $\mathcal{F}$.

4.4. **Rigid Hecke eigensheaves.** The main purpose of this subsection is to establish the existence of Hecke eigensheaves in a special situation. We fix a geometric automorphic datum $(\mathbf{K}_S, \mathcal{K}_S)$.

4.4.1. *Twisted representations.* Let Γ be a group and $\xi \in Z^2(\Gamma, \overline{\mathbb{Q}}_\ell^\times)$ a cocycle such that $\xi_{1,\gamma} = \xi_{\gamma,1} = 1$ for all $\gamma \in \Gamma$. A ξ-twisted representation of Γ is a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space V with automorphisms $T_\gamma : V \to V$, one for each $\gamma \in \Gamma$, such that $T_1 = \mathrm{id}_V$ and

$$T_{\gamma\delta} = \xi_{\gamma,\delta} T_\gamma T_\delta, \forall \gamma, \delta \in \Gamma.$$

Let $\mathrm{Rep}_\xi(\Gamma, \overline{\mathbb{Q}}_\ell)$ be the category of ξ-twisted representations of Γ. This is a $\overline{\mathbb{Q}}_\ell$-linear abelian category which, up to equivalence, only depends on the cohomology class $[\xi] \in \mathrm{H}^2(\Gamma, \overline{\mathbb{Q}}_\ell^\times)$.

4.4.2. Let $\mathcal{E} \in \mathrm{Bun}_G(\mathbf{K}_S)(\overline{k})$ be a relevant point. Let $A = \pi_0(\mathrm{Aut}(\mathcal{E}))$. Since $\mathrm{ev}^*_{S,\mathcal{E}}\mathcal{K}_S$ is trivial on $\mathrm{Aut}^\circ(\mathcal{E})$, it descends to A and gives a cocycle $\xi \in Z^2(A(\overline{k}), \overline{\mathbb{Q}}_\ell)$ satisfying $\xi_{1,a} = \xi_{a,1} = 1$ for all $a \in A(\overline{k})$ whose cohomology class is well-defined (see Remark 2.3.3).

Let Z_0 denote the inverse image of $\prod_{x\in S} L_x Z \cap \mathbf{K}_x$ under the diagonal embedding of Z. Then $Z_0(\overline{k}) \cap \mathbf{K}_S \subset \mathrm{Aut}(\mathcal{E})$. Let $Z_0(\overline{k})^*_\xi$ be the set of ξ-twisted 1-dimensional representations of $Z_0(\overline{k})$ (after restricting the cocycle ξ to $Z_0(\overline{k})$). For $\eta \in Z_0(\overline{k})^*_\xi$, let $\mathrm{Rep}_\xi(A(\overline{k}), \overline{\mathbb{Q}}_\ell)_\eta$ be the full subcategory of $\mathrm{Rep}_\xi(A(\overline{k}), \overline{\mathbb{Q}}_\ell)$ consisting of those ξ-twisted representations of $A(\overline{k})$ whose restriction to $Z_0(\overline{k})$ is the scalar multiplication via η. Then one can decompose $D_G(\mathbf{K}_S, \mathcal{K}_S)$ into a product of categories $D_G(\mathbf{K}_S, \mathcal{K}_S)_\eta$, one for each $\eta \in Z_0(\overline{k})^*_\xi$.

4.4.3. **Theorem.** *We assume* $\mathrm{Bun}_G(\mathbf{K}_S)$ *contains a unique relevant* $\overline{k}$*-point* $\mathcal{E}$*, and let* $A, \xi, Z_0, Z_0(\overline{k})^*_\xi$ *etc. be as in §4.4.2.*

1. *For each* $\eta \in Z_0(\overline{k})^*_\xi$*, the category of perverse sheaves in* $D_G(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)_\eta$ *is equivalent to* $\mathrm{Rep}_\xi(A(\overline{k}), \overline{\mathbb{Q}}_\ell)_\eta$.
2. *Suppose* $\mathrm{Aut}(\mathcal{E})$ *is finite (hence equal to* A*), and the point* $\mathcal{E}$ *is open in* $\mathrm{Bun}_G(\mathbf{K}_S)$*. Suppose further that for some* $\eta \in Z_0(\overline{k})^*_\xi$*,* $\mathrm{Rep}_\xi(A(\overline{k}), \overline{\mathbb{Q}}_\ell)_\eta$ *contains only one irreducible object. Then* $D_G(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)_\eta$ *contains a unique irreducible perverse sheaf* $\mathcal{F}_\eta$ *up to isomorphism, and* $\mathcal{F}_\eta$ *is a Hecke eigensheaf.*

The simplest case of the theorem is when $\mathrm{Aut}(\mathcal{E})$ is trivial. Then $D_G(\overline{k}; \mathbf{K}_S, \mathcal{K}_S)$ contains a unique irreducible perverse sheaf, which is a Hecke eigensheaf. In this case, the geometric automorphic datum $(\mathbf{K}_S, \mathcal{K}_S)$ is strongly rigid if k is a finite field.

4.4.4. *Description of the Hecke eigen local system.* Suppose we are in the situation of Theorem 4.4.3(2), i.e., $\mathcal{E}$ is the unique relevant $\overline{k}$-point in $\mathrm{Bun}_G(\mathbf{K}_S)$, and it is an open point with $A = \mathrm{Aut}(\mathcal{E})$ finite. Let ρ_η be the Hecke eigen $\widehat{G}$-local system associated with the Hecke eigensheaf $\mathcal{F}_\eta$. We shall give a description of $\rho_{\eta,V}$ for $V \in \mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell)$.

For each point $x \in U$ define $\mathfrak{G}_x$ to be the group of G-automorphisms $\mathcal{E}|_{X-\{x\}}$ preserving the $\mathbf{K}_S$-level structures. This is representable by a group ind-scheme. As x varies over U, we get a group ind-schemes $\mathfrak{G}_U$ over U. The $\mathbf{K}_S$-analog of the diagram (4.1) restricted to the open relevant point becomes the diagram

$$\begin{tikzcd} & A\backslash\mathfrak{G}_U/A \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] \arrow[rr, "\pi"] & & U \\ \mathbb{B}A & & \mathbb{B}A & \end{tikzcd}$$

Since $A \subset \mathfrak{G}_x$ for each x, A acts on each fiber $\mathfrak{G}_x$ by left and right translations.

By Remark 2.3.7(1), the character sheaf $\mathcal{K}_S$ on L_S can be obtained from a central extension

$$1 \to C \to \widetilde{L}_S \xrightarrow{v} L_S \to 1$$

where $\widetilde{L}_S$ is connected and C is a finite discrete (necessarily abelian) group scheme over k, and $\chi_C : C \to \overline{\mathbb{Q}}_\ell^\times$ is a character. Concretely $\mathcal{K}_S \cong (v_!\overline{\mathbb{Q}}_\ell)_{\chi_C}$. Let $\widetilde{A}$ and $\widetilde{\mathfrak{G}}_U$ be the base change of the cover $\widetilde{L}_S \to L_S$ along $\mathrm{ev}_{S,\mathcal{E}} : A \to L_S$ (see (2.5)). Then $\widetilde{A} \times \widetilde{A}$ acts on $\widetilde{\mathfrak{G}}_U$ via left and right translations

$$\begin{aligned} \widetilde{A} \times \widetilde{A} \times \widetilde{\mathfrak{G}}_U &\to \widetilde{\mathfrak{G}}_U && (4.4)\\ (a_1, a_2, g) &\mapsto a_1 g a_2^{-1}. && (4.5) \end{aligned}$$

Let $\widetilde{Z}_0 \subset \widetilde{A}$ be the preimage of $Z_0 \subset A$. Via the central extension $\widetilde{A}$ of A, we can interpret $\mathrm{Rep}_\xi(A(\overline{k}), \overline{\mathbb{Q}}_\ell)$ as (ordinary) representations of $\widetilde{A}(\overline{k})$ on which C acts via χ_C. Similarly, a twisted central character $\eta \in Z_0(\overline{k})^*_\xi$ can be viewed as a character of $\widetilde{Z}_0(\overline{k})$ whose restriction to C is χ_C.

The next result gives a concrete description of the Hecke eigen $\widehat{G}$-local systems ρ_η using the geometry of $\widetilde{\mathfrak{G}}_U$ and the $\widetilde{A} \times \widetilde{A}$ action on it. We only need the following maps to state the result

$$U \xleftarrow{\widetilde{\pi}} \widetilde{\mathfrak{G}}_U \xrightarrow{\quad\widetilde{\mathrm{ev}}\quad} \left[\frac{L^+G\backslash LG/L^+G}{\mathrm{Aut}^+}\right]$$

4.4.5. **Proposition.** *In the situation of Theorem 4.4.3(2), let $\mathcal{F}_\eta$ be the Hecke eigensheaf corresponding to the irreducible object $W_\eta \in \mathrm{Rep}_\xi(A(\overline{k}), \overline{\mathbb{Q}}_\ell)_\eta$, viewed as an irreducible representation of $\widetilde{A}(\overline{k})$ as above. Let ρ_η be the corresponding Hecke eigen $\widehat{G}$-local system over $U \otimes_k \overline{k}$. For any $V \in \mathrm{Rep}(\widehat{G}, \overline{\mathbb{Q}}_\ell)$, we have an isomorphism of sheaves over $U \otimes_k \overline{k}$:*

$$\rho_{\eta,V} \cong \mathrm{Hom}_{\widetilde{A}(\overline{k}) \times \widetilde{A}(\overline{k})}(W_\eta \boxtimes W_\eta^\vee, \widetilde{\pi}_! \widetilde{\mathrm{ev}}^* \mathrm{IC}_V). \tag{4.6}$$

Here we are using the left action of $\widetilde{A}(\overline{k}) \times \widetilde{A}(\overline{k})$ on $\widetilde{\pi}_! \widetilde{\mathrm{ev}}^* \mathrm{IC}_V$ induced from the left action defined in (4.4).

4.4.6. *Rationality issue.* Suppose that the unique relevant point $\mathcal{E}$ is defined over k and that the representation W_η can be extended to $\widetilde{A}(\overline{k}) \rtimes \mathrm{Gal}(\overline{k}/k)$, then the Hecke eigensheaf $\mathcal{F}_\eta$ is also defined over k (i.e., it descends to an object in $D_G(\mathbf{K}_S, \chi_S)$), and the Hecke eigen local system ρ_η is a $\widehat{G}$-local system over U. The identity (4.6) now holds over U.

5. Kloosterman sheaves as rigid objects over $\mathbb{P}^1 - \{0, \infty\}$

In this section we review the work [17], in which we used rigid automorphic representations ramified at two places to construct generalizations of Kloosterman sheaves.

5.1. **Kloosterman automorphic data.** Let $k = \mathbb{F}_q$ be a finite field. The curve is $X = \mathbb{P}^1$ with function field $F = k(t)$. Let $S = \{0, \infty\}$. We shall define a geometric automorphic datum for G with respect to S, to be called the *Kloosterman automorphic datum*.

Recall as part of the pinning we have fixed a pair of opposite Borels $B, B^{\mathrm{opp}} \subset G$ with $T = B \cap B^{\mathrm{opp}}$. They determine Iwahori subgroups $\mathbf{I}_\infty \subset L_\infty G$ (preimage of $B \subset G$ under the evaluation map $L_\infty^+ G \to G$) and $\mathbf{I}_0^{\mathrm{opp}} \subset L_0 G$ (preimage of $B^{\mathrm{opp}} \subset G$ under the evaluation map $L_0^+ G \to G$). Let $\mathbf{K}_0 = \mathbf{I}_0^{\mathrm{opp}}$ and $\mathbf{K}_\infty = \mathbf{I}_\infty^+$, the pro-unipotent radical of $\mathbf{I}_\infty$.

Fix a character $\chi : T(k) \to \overline{\mathbb{Q}}_\ell^\times$, and view it as a character of $\mathbf{K}_0(k)$ via $\mathbf{K}_0(k) \to T(k)$. The Kloosterman automorphic datum at $x = 0$ is given by $(\mathbf{I}_0^{\mathrm{opp}}, \chi)$.

The pro-unipotent group $\mathbf{I}_\infty^+$ is generated by the root groups $(L_\infty G)(\widetilde{\alpha})$ of all positive affine roots of the loop group $L_\infty G$. There is a projection

$$\mathbf{K}_\infty = \mathbf{I}_\infty^+ \twoheadrightarrow \prod_{i=0}^{r} (L_\infty G)(\alpha_i) \tag{5.1}$$

onto the product of root groups corresponding to simple affine roots. Each $(L_\infty G)(\alpha_i)$ is isomorphic to $\mathbb{G}_a$ over k. A linear function $\phi : \prod_{i=0}^r (L_\infty G)(\alpha_i) \to k$ is said to be *generic* if it does not vanish on any of the factors $(L_\infty G)(\alpha_i)$. Fix such a generic linear function ϕ and fix a nontrivial additive character ψ of k. The composition $\psi \circ \phi$ gives a character of $\mathbf{I}_\infty^+(k)$ which factors through the quotient (5.1). The Kloosterman automorphic datum at $x = \infty$ is given by $(\mathbf{I}_\infty^+, \psi \circ \phi)$.

A central character compatible with the Kloosterman automorphic datum exists and is unique: its local component at 0 is given by $\chi|_{Z(k)}$ and at 0 is given by $\chi^{-1}|_{Z(k)}$.

5.1.1. **Remark.** The automorphic datum $(\mathbf{I}_\infty^+, \psi\circ\phi)$ picks up those representations of $G(F_\infty)$ that contain nonzero eigenvectors of $\mathbf{K}_\infty$ on which $\mathbf{K}_\infty$ acts through the character $\psi\circ\phi$. These representations are first discovered by Gross and Reeder [15, §9.3], and they call them *simple supercuspidal representations.* When G is simply-connected, any such representation is given by compact induction $\mathrm{ind}_{Z(k)\mathbf{I}_\infty^+(k)}^{G(F_\infty)}(\omega_\infty\boxtimes\psi\circ\phi)$ for some central character $\omega_\infty : Z(k)\to\overline{\mathbb{Q}}_\ell^\times$.

5.1.2. **Theorem** (Gross [13], simple proof by Heinloth-Ngô-Yun [17])**.** (1) *The Kloosterman automorphic datum is strongly rigid.*
(2) *When $\chi = 1$, the local component at 0 of any $((\mathbf{I}_0^{\mathrm{opp}}, 1), (\mathbf{I}_\infty^+, \psi\circ\phi))$-typical automorphic representation is isomorphic to the Steinberg representation of $G(F_0)$.*

Part (1) above is a special case of Theorem 4.4.3, and the automorphism group of the unique relevant point is trivial. By Theorem 4.4.3, we have a Hecke eigen $\widehat{G}$-local system $\mathrm{Kl}_{\widehat{G}}(\chi,\phi)$ over $\mathbb{P}^1 - \{0,\infty\}$ associated with the unique irreducible perverse sheaf in $D_G(\mathbf{K}_S, \chi_S)$.

5.2. Kloosterman sheaves.

5.2.1. *The classical Kloosterman sheaf.* We first recall the definition of Kloosterman sums. Let p be a prime number. Fix a nontrivial additive character $\psi : \mathbb{F}_p \to \overline{\mathbb{Q}}_\ell^\times$. Let $n \geq 2$ be an integer. Then the n-variable Kloosterman sum over $\mathbb{F}_p$ is a function on $\mathbb{F}_p^\times$ whose value at $a\in\mathbb{F}_p^\times$ is

$$\mathrm{Kl}_n(p;a) = \sum_{x_1,\cdots,x_n\in\mathbb{F}_p^\times; x_1x_2\cdots x_n = a} \psi(x_1+\cdots+x_n).$$

These exponential sums arise naturally in the study of automorphic forms for GL_n.

Deligne [5] gave a geometric interpretation of the Kloosterman sum. He considered the following diagram of schemes over $\mathbb{F}_p$

$$\begin{tikzcd} & \mathbb{G}_m^n \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ \mathbb{G}_m & & \mathbb{A}^1 \end{tikzcd}$$

Here π is the morphism of taking the product and σ is the morphism of taking the sum.

5.2.2. **Definition** (Deligne [5])**.** The *Kloosterman sheaf* is

$$\mathrm{Kl}_n := \mathbf{R}^{n-1}\pi_!\sigma^*\mathrm{AS}_\psi,$$

over $\mathbb{P}^1_{\mathbb{F}_p} - \{0,\infty\}$. Here AS_ψ is the Artin-Schreier sheaf as in Example 2.3.6.

The relationship between the local system Kl_n and the Kloosterman sum $\mathrm{Kl}_n(p;a)$ is explained by the following identity

$$\mathrm{Kl}_n(p;a) = (-1)^{n-1}\mathrm{Tr}(\mathrm{Frob}_a, (\mathrm{Kl}_n)_a).$$

Here Frob_a is the geometric Frobenius operator acting on the geometric stalk $(\mathrm{Kl}_n)_a$ of Kl_n at $a\in\mathbb{G}_m(\mathbb{F}_p) = \mathbb{F}_p^\times$.

In [5, Théorème 7.4, 7.8], Deligne proved:

(1) Kl_n is a local system of rank n.
(2) Kl_n is tamely ramified at 0, and the monodromy is unipotent with a single Jordan block.
(3) Kl_n is totally wild at ∞ (i.e., the wild inertia at ∞ has no nonzero fixed vector on the stalk of Kl_n), and the Swan conductor $\mathrm{Sw}_\infty(\mathrm{Kl}_n) = 1$.

The main results of [17] can be summarized as follows.

5.2.3. **Theorem** (Heinloth-Ngô-Yun [17]). *The $\widehat{G}$-local system $\mathrm{Kl}_{\widehat{G}}(\chi,\phi)$ constructed as the Hecke eigen $\widehat{G}$-local system associated with the Kloosterman automorphic datum satisfies the following properties.*

(1) $\mathrm{Kl}_{\widehat{G}}(\chi,\phi)$ *is tame at* 0. *A generator of* $\mathcal{I}_0^t$ *maps to an element in* $\widehat{G}$ *with semisimple part given by* χ*, viewed as an element in* $\widehat{T}$. *When* $\chi = 1$*, a generator of* $\mathcal{I}_0^t$ *maps to a regular unipotent element in* $\widehat{G}$.

(2) *The local monodromy of* $\mathrm{Kl}_{\widehat{G}}(\chi,\phi)$ *at* ∞ *is a* simple wild parameter *in the sense of Gross and Reeder* [15, §5]. *For more details see §5.2.5.*

(3) *Assume* $\chi = 1$. *Then the global geometric monodromy group of* $\mathrm{Kl}_{\widehat{G}}(1,\phi)$ *is a connected almost simple subgroup of* $\widehat{G}$ *of types given by the following table*

$\widehat{G}$	$\widehat{G}^{\mathrm{geom}}_{\mathrm{Kl}_{\widehat{G}}(1,\phi)}$	condition on $\mathrm{char}(k)$
A_{2n}	A_{2n}	$p>2$
A_{2n-1}, C_n	C_n	$p>2$
B_n, D_{n+1} $(n\geq 4)$	B_n	$p>2$
E_7	E_7	$p>2$
E_8	E_8	$p>2$
E_6, F_4	F_4	$p>2$
B_3, D_4, G_2	G_2	$p>3$

5.2.4. **Remark.** For $\widehat{G}$ of type A_{n-1} (resp. C_n), the Kloosterman sheaf $\mathrm{Kl}_{\widehat{G}}(1,\phi)$ is essentially the same as Kl_n (resp. Kl_{2n}) of Deligne. When $\widehat{G}$ is of type B or G_2, $\mathrm{Kl}_{\widehat{G}}(1,\phi)$ were constructed by Katz in [19] by different methods (as special cases of hypergeometric sheaves). The above theorem treats all $\widehat{G}$ uniformly, and in particular gives the first construction of motivic local systems with geometric monodromy group F_4, E_7 and E_8.

5.2.5. *Local monodromy at* ∞. Let us explain in more detail what a *simple wild parameter* looks like, following Gross and Reeder [15, Proposition 5.6]. Assume $p=\mathrm{char}(k)$ does not divide $\#W$ (W is the Weyl group of G). Let $\rho|_{\mathcal{I}_\infty}:\mathcal{I}_\infty\to\widehat{G}(\overline{\mathbb{Q}}_\ell)$ be the local monodromy of $\mathrm{Kl}_{\widehat{G}}(\chi,\phi)$ at ∞. Then up to conjugacy in $\widehat{G}$, the wild inertia $\mathcal{I}_\infty^w$ has image in $\widehat{T}[p]$, the p-torsion part of the dual maximal torus $\widehat{T}$. The image of $\mathcal{I}_\infty$ must normalize the image of $\mathcal{I}_\infty^w$, and in this case it must normalize the whole torus $\widehat{T}$. Therefore we have a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{I}_\infty^w \arrow[r] \arrow[d, "\rho|_{\mathcal{I}_\infty^w}"] & \mathcal{I}_\infty \arrow[r] \arrow[d, "\rho|_{\mathcal{I}_\infty}"] & \mathcal{I}_\infty^t \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \widehat{T} \arrow[r] & N_{\widehat{G}}(\widehat{T}) \arrow[r] & W \arrow[r] & 1
\end{tikzcd}$$

The image of $\mathcal{I}_\infty^t$ in W is the cyclic group generated by a Coxeter element $\mathrm{Cox}\in W$, whose order is the Coxeter number h of G. The image $\rho(\mathcal{I}_\infty^w)$ is a $\mathbb{F}_p$-vector space equipped with the action of Cox. In fact $\rho(\mathcal{I}_\infty^w)\cong\mathbb{F}_p[\zeta_h]$, the extension of $\mathbb{F}_p$ by adjoining h^{th} roots of unity, and the Coxeter element acts by multiplication by a primitive h^{th} root of unity.

When $p\mid\#W$, a simple wild parameter can be more complicated. For example, when $\widehat{G}=\mathrm{PGL}_2$ and $p=2$, the image $\rho(\mathcal{I}_\infty)$ is isomorphic to the alternating group A_4 embedded in $\mathrm{PGL}_2(\overline{\mathbb{Q}}_\ell)=\mathrm{SO}_3(\overline{\mathbb{Q}}_\ell)$ as the symmetry of a regular tetrahedron.

5.2.6. *More general Kloosterman sheaves.* In [37] we give further generalizations of Kloosterman sheaves. We replace $\mathbf{I}_0^{\mathrm{opp}}$ by a more general parahoric level structure $\mathbf{P}_0^{\mathrm{opp}}$, and accordingly $\mathbf{I}_\infty^+$

is replaced by a $\mathbf{P}_\infty^+$, the pro-unipotent radical of a parahoric of the same type. The geometric automorphic datum at 0 is again given by a multiplicative character χ of the reductive quotient of $\mathbf{I}_0^{\mathrm{opp}}$. The geometric automorphic datum at ∞ is given by a generic linear function ϕ from $\mathbf{P}_\infty^+$ (with a suitable notion of genericity). The representations of $G(F_\infty)$ picked up by $(\mathbf{P}_\infty^+, \psi\circ\phi)$ are the *epipelagic representations* discovered by Gross, Reeder and Yu [31]. One new feature of this generalization is that when ϕ varies in $V^{*,\circ}$ (the parameter space of generic linear functions on $\mathbf{P}_\infty^+$), the corresponding generalized Kloosterman sheaves "glue" together to give a $\widehat{G}$-local system over $V^{*,\circ}\times(\mathbb{P}^1-\{0,\infty\})$.

6. Rigid objects over $\mathbb{P}^1-\{0,1,\infty\}$

In this section, we review the work [36], in which we use rigid automorphic representations to construct local systems on $\mathbb{P}^1_{\mathbb{Q}}-\{0,1,\infty\}$. These local systems are the key objects that lead to the answer to Serre's question and the solution of the inverse Galois problem for certain finite simple groups of exceptional Lie type.

6.1. **The automorphic data.** Let k be a field with $\mathrm{char}(k)\neq 2$. Let $X=\mathbb{P}^1_k$ and $S=\{0,1,\infty\}$. Assume that the longest element w_0 in the Weyl group W of G acts by inversion on T. Equivalently, this means that the Chevalley involution of G is inner. Recall that a Chevalley involution of G is an involution τ such that $\dim G^\tau$ has the minimal possible dimension, namely $\#\Phi^+$ (the number of positive roots of G). All Chevalley involutions are conjugate to each other under $G^{\mathrm{ad}}(\overline{k})$.

When the Chevalley involution of G is not inner, i.e., G is of type A_n ($n\geq 2$), D_{2n+1} or E_6, one should consider a quasi-split form of G over the function field $F=k(t)$ ramified at 0 and ∞. We do not discuss it here.

6.1.1. *A parahoric subgroup.* Up to conjugacy, there is a unique parahoric subgroup $\mathbf{P}\subset L_0G$ such that its reductive quotient $L_{\mathbf{P}}$ is isomorphic to the fixed point subgroup G^τ of a Chevalley involution. For example, we can take $\mathbf{P}$ to be the parahoric subgroup corresponding to the facet containing the element $\rho^\vee/2$ in the T-apartment of the building of L_0G ($\rho^\vee$ is half the sum of positive coroots of G).

The Dynkin diagram of the reductive quotient $L_{\mathbf{P}}\cong G^\tau$ of $\mathbf{P}$ is obtained by removing one or two nodes from the extended Dynkin diagram of G. We tabulate the type of $L_{\mathbf{P}}$ and the nodes to be removed in each case.

G	$L_{\mathbf{P}}$	nodes to be removed
B_{2n}	$B_n\times D_n$	the $(n+1)$-th counting from the short node
B_{2n+1}	$B_n\times D_{n+1}$	the $(n+1)$-th counting from the short node
C_n	$A_{n-1}\times\mathbb{G}_m$	the two ends
D_{2n}	$D_n\times D_n$	the middle node
E_7	A_7	the end of the leg of length 1
E_8	D_8	the end of the leg of length 2
F_4	$A_1\times C_3$	second from the long node end
G_2	$A_1\times A_1$	middle node

6.1.2. **Lemma.** *If G is not of type C, then $L^{\mathrm{sc}}_{\mathbf{P}}\to L_{\mathbf{P}}$ is a double cover (i.e., kernel is μ_2).*

Even if G is of type C_n, $L_{\mathbf{P}}\cong\mathrm{GL}_n$ still admits a unique nontrivial double cover. Therefore, in all cases, there is a canonical nontrivial double cover $v:\widetilde{L}_{\mathbf{P}}\to L_{\mathbf{P}}$. In particular,

$$\mathcal{K}_0:=(v_!\overline{\mathbb{Q}}_\ell)_{\mathrm{sgn}}$$

is a rank one character sheaf on $L_{\mathbf{P}}$ (here sgn denotes the nontrivial character of $\ker(v)(k) = \mu_2(k)\{\pm 1\}$, since $\mathrm{char}(k) \neq 2$). When k is a finite field, we have an exact sequence

$$1 \to \mu_2(k) \to \widetilde{L}_{\mathbf{P}}(k) \to L_{\mathbf{P}}(k) \to \mathrm{H}^1(k, \mu_2) = \{\pm 1\}.$$

The character χ corresponding to $\mathcal{K}_0$ is given by the last arrow above.

6.1.3. *The automorphic datum.* Let $\mathbf{P}_0 \subset L_0 G$ be the standard parahoric subgroup of the type defined in §6.1.1. Let $\mathbf{P}^{\mathrm{opp}}_\infty \subset L_\infty G$ be the parahoric subgroup of the same type but contains the Iwahori subgroup $\mathbf{I}^{\mathrm{opp}}_\infty$ (preimage of B^{opp} under the evaluation map $L^+_\infty G \to G$). Let $\mathbf{I}_1 \subset L_1 G$ be the standard Iwahori subgroup. We consider the geometric automorphic datum given by

$$\begin{array}{rcl}(\mathbf{K}_0, \mathcal{K}_0) & = & (\mathbf{P}_0, \mathcal{K}_0);\\ (\mathbf{K}_1, \mathcal{K}_1) & = & (\mathbf{I}_1, \overline{\mathbb{Q}}_\ell);\\ (\mathbf{K}_\infty, \mathcal{K}_\infty) & = & (\mathbf{P}^{\mathrm{opp}}_\infty, \overline{\mathbb{Q}}_\ell).\end{array}$$

The central character compatible with $(\mathbf{K}_S, \chi_S)$ above is unique, and it exists if and only if $\chi|_{Z(k)} = 1$, which can always be achieved by passing to a quadratic extension k'/k.

The main technical result of [36] is the following.

6.1.4. **Theorem** ([36]). *Let k be a field with $\mathrm{char}(k) \neq 2$. Assume G is either simply-laced or of type G_2. Also assume that k contains $\sqrt{-1}$ when G is of type A_1, D_{4n+2} or E_7. Then there are exactly $\#Z(\bar{k})$ isomorphism classes of irreducible perverse sheaves in the category $D_G(\bar{k}; \mathbf{K}_S, \mathcal{K}_S)$, all of which can be defined over k. Each of these irreducible perverse sheaves is a Hecke eigensheaf, and all of them give the same Hecke eigen $\widehat{G}$-local system $\rho \in \mathrm{Loc}_{\widehat{G}}(U)$.*

Note that the condition put on G in the above theorem limits G to be simply-connected of type A_1, D_{2n}, E_7, E_8 and G_2. There is a version of the above theorem for groups of all types; however, the automorphic datum (6.1) will no longer be weakly rigid in general.

6.1.5. We briefly explain how Theorem 6.1.4 follows from Theorem 4.4.3(2). We first exhibit an open point in $\mathrm{Bun}_G(\mathbf{K}_S)$ which is relevant. The moduli stack $\mathrm{Bun}_G(\mathbf{P}_0, \mathbf{P}^{\mathrm{opp}}_\infty)$ contains an open substack isomorphic to $\mathbb{B}L_{\mathbf{P}}$, the classifying stack of $L_{\mathbf{P}}$. The preimage of $\mathbb{B}L_{\mathbf{P}}$ in $\mathrm{Bun}_G(\mathbf{K}_S)$ is isomorphic to $L_{\mathbf{P}}\backslash G/B$. Now a general result of Springer [34, Corollary 4.3(i)] says that any symmetric subgroup of G acts on the flag variety of G with an open orbit. Let $O \subset G/B$ denote the unique open $L_{\mathbf{P}}$-orbit. We thus get an open point

$$j : L_{\mathbf{P}}\backslash O \hookrightarrow \mathrm{Bun}_G(\mathbf{K}_S).$$

When G is either simply-laced or of type G_2, this turns out to be the unique relevant point.

The stabilizer A_u of $L_{\mathbf{P}}$ on any point $u \in O$ is canonically isomorphic to $T[2]$. Taking the preimage of A_u in the double cover $\widetilde{L}_{\mathbf{P}}$ we get a central extension

$$1 \to \mu_2 \to \widetilde{A}_u \to T[2] \to 1.$$

The center of $\widetilde{A}_u(\bar{k})$ is exactly the preimage $\widetilde{Z}(\bar{k})$ in $\widetilde{A}_u(\bar{k})$ of $Z(\bar{k}) \subset T[2](\bar{k})$. By Stone-von Neumann theorem, for any central character $\eta : \widetilde{Z}(\bar{k}) \to \overline{\mathbb{Q}}_\ell^\times$ which is nontrivial on $\mu_2(\bar{k}) = \ker(\widetilde{A}_u(\bar{k}) \to T[2](\bar{k}))$, there is a unique irreducible representation W_η of $\widetilde{A}_u(\bar{k})$ with central character η. This W_η then gives an irreducible local system on the preimage of the open point $L_{\mathbf{P}}\backslash O$ in $\mathrm{Bun}_G(\mathbf{K}^+_S)$, whose extension to $\mathrm{Bun}_G(\mathbf{K}^+_S)$ by zero gives an object $\mathcal{F}_\eta \in D_G(\bar{k}; \mathbf{K}_S, \mathcal{K}_S)$. The $\mathcal{F}_\eta$ are exactly the irreducible perverse sheaves in $D_G(\bar{k}; \mathbf{K}_S, \mathcal{K}_S)$, and they turn out to be Hecke eigensheaves.

The following theorem summarizes the local and global monodromy of the $\widehat{G}$-local system ρ.

6.1.6. **Theorem** ([36]). *Let G and ρ be as in Theorem 6.1.4.*

(1) *The local system ρ is tame.*
(2) *A topological generator of $\mathcal{I}_1^t$ maps to a regular unipotent element in $\widehat{G}$.*
(3) *A topological generator of $\mathcal{I}_\infty^t$ maps to a unipotent element in $\widehat{G}$ that is neither trivial nor regular if G is not of type A_1; in case G is of type A_1, a topological generator of $\mathcal{I}_\infty^t$ maps also to a regular unipotent element in $\widehat{G}$.*
(4) *A topological generator of $\mathcal{I}_0^t$ maps to an element in $\widehat{G}$ whose semisimple part is a Chevalley involution in $\widehat{G}$.*
(5) *When G is of type A_1, E_7, E_8 and G_2, the global geometric monodromy of ρ is Zariski dense in $\widehat{G}$. When G is of type D_{2n}, the Zariski closure of the global geometric monodromy of $\mathcal{E}$ contains $\mathrm{SO}_{4n-1} \subset \mathrm{PSO}_{4n} = \widehat{G}$ of $n \geq 3$, and contains $G_2 \subset \mathrm{PSO}_8$ if $n = 2$.*

6.2. **Applications.** By a descent argument (using rigidity), we have the following strengthening of Theorem 6.1.4.

6.2.1. **Theorem** ([36]). *Let k be a prime field with $\mathrm{char}(k) \neq 2$ (i.e., $\mathbb{F}_p$ for p an odd prime or $\mathbb{Q}$). Then the eigen local system ρ can be defined over k. Moreover, the image of ρ can be conjugated to $\widehat{G}(\mathbb{Q}_\ell)$ inside $\widehat{G}(\overline{\mathbb{Q}}_\ell)$.*

6.2.2. *Application to the construction of motives.* Assume G is of type A_1, E_7, E_8 or G_2. Applying the above theorem to $k = \mathbb{Q}$, we get a $\widehat{G}$-local systems $\rho : \pi_1(U_{\mathbb{Q}}) \to \widehat{G}(\mathbb{Q}_\ell)$ whose geometric monodromy is Zariski dense. For each $\mathbb{Q}$-point $a \in U(\mathbb{Q}) = \mathbb{Q} - \{0, 1\}$, restricting ρ to the point $a = \mathrm{Spec}\,\mathbb{Q}$ gives a continuous Galois representation

$$\rho_a : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \widehat{G}(\mathbb{Q}_\ell). \tag{6.1}$$

By Proposition 4.4.5, one sees that for each $V \in \mathrm{Rep}(\widehat{G}, \mathbb{Q}_\ell)$, ρ_V is obtained as part of the middle dimensional cohomology of some family of varieties over U. Using this fact, it can be shown that each ρ_a is obtained from motives over $\mathbb{Q}$ (if G is type E_8 or G_2) or $\mathbb{Q}(i)$ (if G is of type A_1 or E_7).

6.2.3. **Theorem** ([36]). *Assume G is of type A_1, E_7, E_8 or G_2. There are infinitely many $a \in \mathbb{Q} - \{0, 1\}$ such that the ρ_a's are mutually non-isomorphic and all have Zariski dense image in $\widehat{G}$. Consequently, there are infinitely many motives over $\mathbb{Q}$ (if G is type E_8 or G_2) or $\mathbb{Q}(i)$ (if G is of type A_1 or E_7) whose ℓ-adic motivic Galois group is isomorphic to $\widehat{G}$ for any prime ℓ.*

This result then gives an affirmative answer to the ℓ-adic analog of Serre question (see §1.3.2) for motivic Galois groups of type E_7, E_8 and G_2. The case of G_2 was settled earlier by Dettweiler and Reiter [8], using Katz's algorithmic construction of rigid local systems. Our local system ρ in the case $G = G_2$ is the same as Dettweiler and Reiter's.

6.2.4. *Application to the inverse Galois problem.* Let ℓ be a prime number. To emphasize on the dependence on ℓ, we denote the Galois representation ρ_a in (6.1) by $\rho_{a,\ell}$. To solve the inverse Galois problem for the groups $\widehat{G}(\mathbb{F}_\ell)$, we would like to choose $a \in \mathbb{Q} - \{0, 1\}$ such that $\rho_{a,\ell}$ has image in $\widehat{G}(\mathbb{Z}_\ell)$ (which is always true up to conjugation), and its reduction modulo ℓ is surjective. This latter condition is hard to satisfy even if we know that the image of $\rho_{a,\ell}$ is Zariski dense in $\widehat{G}(\mathbb{Q}_\ell)$.

To proceed, let us consider the Betti version of Theorem 6.1.4 and Theorem 6.2.1. Namely we consider the base field $k = \mathbb{C}$ and talk about sheaves in $\mathbb{Q}$-vector spaces on the various complex algebraic moduli stacks. The same argument gives topological local system ρ^{top} over $U_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}} - \{0,1,\infty\}$

$$\rho^{\mathrm{top}} : \pi_1^{\mathrm{top}}(U_{\mathbb{C}}) \to \widehat{G}(\mathbb{Q})$$

whose image is Zariski dense. Since $\pi_1^{\mathrm{top}}(U_{\mathbb{C}})$ is finitely generated (in fact generated by two elements), the image of ρ^{top} lies in $\widehat{G}(\mathbb{Z}_\ell)$ for almost all primes ℓ. Therefore it makes sense to reduce modulo ℓ for large primes ℓ and we get

$$\overline{\rho}_\ell^{\mathrm{top}} : \pi_1^{\mathrm{top}}(U_{\mathbb{C}}) \to \widehat{G}(\mathbb{F}_\ell).$$

A deep theorem of Matthews, Vaserstein and Weisfeiler [29, Theorem in the Introduction] (see also Nori [30, Theorem 5.1]) says that $\overline{\rho}_\ell^{\mathrm{top}}$ is surjective for sufficiently large ℓ, when $\widehat{G}$ is simply-connected. This is the case when G is of type E_8 and G_2. Using the comparison between Betti cohomology and ℓ-adic cohomology, we conclude that for general $a \in \mathbb{Q} - \{0,1\}$ (general in the sense of Hilbert irreducibility), the reduction $\overline{\rho}_{a,\ell}$ of $\rho_{a,\ell}$ is also onto $\widehat{G}(\mathbb{F}_\ell)$. This solves the inverse Galois problem for $E_8(\mathbb{F}_\ell)$ and $G_2(\mathbb{F}_\ell)$ for sufficiently large primes ℓ (without an effective bound).

When G is of type A_1 or E_7, $\widehat{G}$ is the adjoint form. In this case, the result in [29] says that for sufficiently large prime ℓ, the image of $\overline{\rho}_\ell^{\mathrm{top}}$ contains the image of $\widehat{G}^{\mathrm{sc}}(\mathbb{F}_\ell) \to \widehat{G}(\mathbb{F}_\ell)$. We deduce that the same is true for $\rho_{a,\ell}$ for general $a \in \mathbb{Q} - \{0,1\}$.

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