REPRESENTATION THEORY
An Electronic Journal of the American Mathematical Society
Volume 21, Pages 277–321 (September 14, 2017)
http://dx.doi.org/10.1090/ert/500

$\mathbf{Z}/m$-GRADED LIE ALGEBRAS AND PERVERSE SHEAVES, I

GEORGE LUSZTIG AND ZHIWEI YUN

ABSTRACT. We give a block decomposition of the equivariant derived category arising from a cyclically graded Lie algebra. This generalizes certain aspects of the generalized Springer correspondence to the graded setting.

CONTENTS

INTRODUCTION

0.1. Let $\mathbf{k}$ be an algebraically closed field of characteristic $p \geq 0$. We fix an integer $m > 0$ such that $m < p$ whenever $p > 0$ and we write $\mathbf{Z}/m$ instead of $\mathbf{Z}/m\mathbf{Z}$. For $n \in \mathbf{Z}$, let $\underline{n}$ denote the image of n in $\mathbf{Z}/m$.

We also fix G, a semisimple simply connected algebraic group over $\mathbf{k}$ and a $\mathbf{Z}/m$-grading $\mathfrak{g} = \oplus_{i \in \mathbf{Z}/m} \mathfrak{g}_i$ (see 0.11) for the Lie algebra $\mathfrak{g}$ of G; we shall assume that either $p = 0$ or that p is so large relative to G, that we can operate with $\mathfrak{g}$ as if p was 0.

For any integer d invertible in $\mathbf{k}$ let $\mu_d = \{z \in \mathbf{k}^*; z^d = 1\}$. The $\mathbf{Z}/m$-grading on $\mathfrak{g}$ is the same as an action of μ_m on G or a homomorphism $\tilde{\vartheta} : \mu_m \to \mathrm{Aut}(G)$. ($\tilde{\vartheta}$ induces a homomorphism $\tilde{\theta} : \mu_m \to \mathrm{Aut}(\mathfrak{g})$ and for $i \in \mathbf{Z}/m$ we have $\mathfrak{g}_i = \{x \in \mathfrak{g}; \tilde{\theta}(z)x = z^i x \quad \forall z \in \mu_m\}$.) Let $G_{\underline{0}} = \{g \in G; g\tilde{\vartheta}(z) = \tilde{\vartheta}(z)g \quad \forall z \in \mu_m\}$, be a connected reductive subgroup of G with Lie algebra $\mathfrak{g}_{\underline{0}}$. For any $i \in \mathbf{Z}/m$, the Ad-action of $G_{\underline{0}}$ on $\mathfrak{g}$ leaves stable $\mathfrak{g}_i$ and its closed subset $\mathfrak{g}_i^{nil} := \mathfrak{g}_i \cap \mathfrak{g}^{nil}$. (Here $\mathfrak{g}^{nil}$ is the variety of nilpotent elements in $\mathfrak{g}$.)

Received by the editors October 12, 2016, and, in revised form, June 23, 2017.
2010 *Mathematics Subject Classification.* Primary 20G99.
The first author was supported by NSF grant DMS-1566618.
The second author was supported by NSF grant DMS-1302071 and the Packard Foundation.

We are interested in studying the equivariant derived categories (see 0.11) $\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_i)$, $\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_i^{nil})$. More specifically, we would like to classify $G_{\underline{0}}$-equivariant simple perverse sheaves with support in $\mathfrak{g}_i^{nil}$ and (in the case where $p > 0$) their Fourier-Deligne transform. The simple perverse sheaves in $\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_i^{nil})$ are in bijection with the pair $(\mathcal{O}, \mathcal{L})$, where $\mathcal{O}$ is a nilpotent $G_{\underline{0}}$-orbit in $\mathfrak{g}_i$ and $\mathcal{L}$ is (the isomorphism class of) an irreducible $G_{\underline{0}}$-equivariant local system on $\mathcal{O}$. (The pair $(\mathcal{O}, \mathcal{L})$ gives rise to the simple perverse sheaf P with support equal to the closure of $\mathcal{O}$ and with $P|_{\mathcal{O}} = \mathcal{L}[\dim \mathcal{O}]$.) We denote the set of such $(\mathcal{O}, \mathcal{L})$ by $\mathcal{I}(\mathfrak{g}_i)$. This is a finite set, since the $G_{\underline{0}}$-action on $\mathfrak{g}_i^{nil}$ has only finitely many orbits. Alternatively, if we choose $e \in \mathcal{O}$, then the local system $\mathcal{L}$ corresponds to an irreducible representation of $\pi_0(G_{\underline{0}}(e))$ (see 0.11), where $G_{\underline{0}}(e)$ is the stabilizer of e under $G_{\underline{0}}$.

There are many $\mathbf{Z}/m$-graded Lie algebras which appear in nature.

0.2. In this subsection we assume that $m = 2$ and $\mathbf{k} = \mathbf{C}$. Then the $\mathbf{Z}/2$-grading $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (with $\mathfrak{k} = \mathfrak{g}_{\underline{0}}$, $\mathfrak{p} = \mathfrak{g}_{\underline{1}}$) has been extensively studied in connection with the theory of symmetric spaces and the representation theory of real semisimple groups. In particular, the nilpotent $G_{\underline{0}}$-orbits on $\mathfrak{p}$ are known to be in bijection with the nilpotent orbits in the Lie algebra of a real form of G determined by the $\mathbf{Z}/2$-grading (Kostant and Sekiguchi).

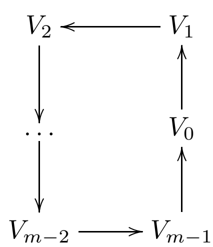
0.3. Another class of examples comes from cyclic quivers. In this subsection we assume that $m \geq 2$. We consider the simplest such example where V is a $\mathbf{k}$-vector space equipped with a $\mathbf{Z}/m$-grading $V = \oplus_{i \in \mathbf{Z}/m} V_i$ (see 0.11) and $G = SL(V)$ with the $\mathbf{Z}/m$-grading given by

$$\mathfrak{g}_i = \{T \in \mathfrak{g} = \mathfrak{sl}(V); T(V_j) \subset V_{j+i} \quad \forall j \in \mathbf{Z}/m\}.$$

In this case we have $G_{\underline{0}} = S(\prod_{i \in \mathbf{Z}/m} GL(V_i))$, the intersection of $SL(V)$ with the Levi subgroup $\prod_i GL(V_i)$ of a parabolic subgroup of $GL(V)$. The subspace $\mathfrak{g}_{\underline{1}}$ is

(a) $$\oplus_{i \in \mathbf{Z}/m} \operatorname{Hom}(V_i, V_{i+1}).$$

We may consider a quiver Q with m vertices indexed by $\mathbf{Z}/m$ and an arrow $i \mapsto i+1$ for each $i \in \mathbf{Z}/m$,



Then $\mathfrak{g}_{\underline{1}}$ is the space of representations of Q where we put V_i at the vertex i.

More generally, if G is a classical group, then the $G_{\underline{0}}$-action on $\mathfrak{g}_{\underline{1}}$ can be interpreted in terms of a cyclic quiver with some extra structure (see 9.5 for the case where G is a symplectic group).

0.4. In this subsection we forget the $\mathbf{Z}/m$-grading. Instead of the action of $G_{\underline{0}}$ on $\mathfrak{g}_i$ and $\mathfrak{g}_i^{nil}$ we consider the adjoint action of G on $\mathfrak{g}$ and on $\mathfrak{g}^{nil}$. Let $\mathcal{I}(\mathfrak{g})$ be the set of pairs $(\mathcal{O}, \mathcal{L})$ where $\mathcal{O}$ is a G-orbit on $\mathfrak{g}^{nil}$ and $\mathcal{L}$ is an irreducible G-equivariant local system on $\mathcal{O}$ (up to isomorphism). From the results on the generalized Springer theory in [L1] we have a canonical decomposition

(a) $$\mathcal{I}(\mathfrak{g}) = \sqcup_{(L,C)}\mathcal{I}(\mathfrak{g})_{(L,C)},$$

where (L, C) runs over the G-conjugacy classes of data L, C with L a Levi subgroup of a parabolic subgroup of G and C an L-equivariant cuspidal perverse sheaf on the nilpotent cone of the Lie algebra of L. (Actually, the results of [L1] are stated for unipotent elements in G instead of nilpotent elements in $\mathfrak{g}$.) We call (a) the *block decomposition* of $\mathcal{I}(\mathfrak{g})$.

Let $P(\mathfrak{g}^{nil})$ be the subcategory of $\mathcal{D}(\mathfrak{g}^{nil})$ consisting of complexes whose perverse cohomology sheaves are G-equivariant. Using (a) and [L3, (7.3.1)], we see that we have a direct sum decomposition

(b) $$P(\mathfrak{g}^{nil}) = \oplus_{(L,C)}P(\mathfrak{g}^{nil})_{(L,C)},$$

where (L, C) is as in (a). We call (b) the *block decomposition* of $P(\mathfrak{g}^{nil})$. In [RR] it is shown that the following variant of (b) holds: we have a direct sum decomposition

(c) $$\mathcal{D}_G(\mathfrak{g}^{nil}) = \oplus_{(L,C)}\mathcal{D}_G(\mathfrak{g}^{nil})_{(L,C)},$$

where (L, C) is as in (a). We call (c) the *block decomposition* of $\mathcal{D}_G(\mathfrak{g}^{nil})$.

In this paper we find a $\mathbf{Z}/m$-graded analogue of this (ungraded) block decomposition.

0.5. We fix ζ, a primitive m-th root of 1 in $\mathbf{k}$ and we set $\vartheta = \tilde{\vartheta}(\zeta) : G \to G$, $\theta = \tilde{\theta}(\zeta) : \mathfrak{g} \to \mathfrak{g}$. Then for $i \in \mathbf{Z}/m$ we have $\mathfrak{g}_i = \{x \in \mathfrak{g}; \theta(x) = \zeta^i x\}$.

Let $\eta \in \mathbf{Z} - \{0\}$. We consider systems $(M, \mathfrak{m}_*, \tilde{C})$, where

$$M = \{g \in G; \mathrm{Ad}(\tau)\vartheta g = g\}$$

for some semisimple element of finite order $\tau \in G_{\underline{0}}$, $\mathfrak{m}_* = \{\mathfrak{m}_N\}_{N\in\mathbf{Z}}$ is a $\mathbf{Z}$-grading of the Lie algebra $\mathfrak{m}$ of M (see 0.11) such that $\mathfrak{m}_N \subset \mathfrak{g}_{\underline{N}}$ for all N, M_0 is the closed connected subgroup of M with Lie algebra $\mathfrak{m}_0$ and $\tilde{C}$ is an M_0-equivariant cuspidal perverse sheaf on $\mathfrak{m}_\eta$. We will review the notion of M_0-equivariant cuspidal perverse sheaf (already defined in [L4]) on $\mathfrak{m}_\eta$ in 1.2. Such a system $(M, \mathfrak{m}_*, \tilde{C})$ is said to be *admissible* if a certain technical condition involving the group of components of the center of M is satisfied (see 3.1).

Let $\underline{\mathfrak{T}}_\eta$ be the set of admissible systems up to $G_{\underline{0}}$-conjugacy. The following result is proved in 7.9.

Theorem 0.6. *There is a canonical direct sum decomposition of $\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_{\underline{\eta}}^{nil})$ into full subcategories*

$$\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_{\underline{\eta}}^{nil}) = \oplus_{(M,\mathfrak{m}_*,\tilde{C})\in\underline{\mathfrak{T}}_\eta}\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_{\underline{\eta}}^{nil})_{(M,\mathfrak{m}_*,\tilde{C})}$$

indexed by $\underline{\mathfrak{T}}_\eta$.

In particular, any simple perverse sheaf in $\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_{\underline{\eta}}^{nil})$ belongs to a well-defined block $\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_{\underline{\eta}}^{nil})_{(M,\mathfrak{m}_*,\tilde{C})}$. This gives a map

$$\Psi : \mathcal{I}(\mathfrak{g}_{\underline{\eta}}) \to \underline{\mathfrak{T}}_\eta.$$

In fact, we will first establish the map Ψ in 3.5 and then prove the theorem in 7.9, using a key calculation in Proposition 6.4.

We also show in 3.9 and 7.8 that both the indexing set $\underline{\mathfrak{T}}_\eta$ and the blocks $\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_{\underline{\eta}}^{nil})_\xi$ (for $\xi \in \underline{\mathfrak{T}}_\eta$) only depend on the image $\underline{\eta} \in \mathbf{Z}/m$ and not on the integer η.

Note that in the case where $m = 1$, the theorem can be deduced from 0.4(a). On the other hand, for large m, a $\mathbf{Z}/m$-grading on $\mathfrak{g}$ is the same as a $\mathbf{Z}$-grading, so that in this case the theorem can be deduced from the results of [L4]. Thus, the result about block decomposition in this paper generalizes results in [L1] and [L4].

0.7. As an explicit example, let us consider the case where $G = SL_n(\mathbf{k}), \eta = 1$. In the ungraded case, blocks are in bijection with pairs (d, χ) where d is a divisor of n and $\chi : \mu_d \to \bar{\mathbf{Q}}_l^*$ is a primitive character. (See [L1].) To d we attach the subgroup $M = S(GL_d^{n/d})$ (a Levi subgroup of a parabolic subgroup) and χ gives a cuspidal perverse sheaf C_χ with support equal to the nilpotent cone of the Lie algebra of M. Now in the $\mathbf{Z}/m$-graded case, we have $G = SL(V)$, $V = \oplus_{i\in\mathbf{Z}/m} V_i$ as in 0.3, and we identify $\mathfrak{g}_{\underline{1}}$ with $\oplus_i \operatorname{Hom}(V_i, V_{i+1})$. In this case, the set of blocks $\underline{\mathfrak{T}}_1$ has a similar explicit description. We have a natural bijection

(a) $$\underline{\mathfrak{T}}_1 \leftrightarrow \{(d, f, \chi)\}/\sim .$$

Here the right hand side is the set of equivalence classes of triples (d, f, χ) where (d, χ) is as in the ungraded case and $f : \{1, 2, \dots, n/d\} \to \mathbf{Z}/m$ is a map such that

(b) $$\sharp\{(b, y) \in \mathbf{Z} \times \mathbf{Z}; 1 \le b \le n/d, 0 \le y \le d-1, f(b) + \underline{y} = i\} = \dim V_i$$

for all $i \in \mathbf{Z}/m$. Two triples (d, f, χ) and (d', f', χ') are equivalent if and only if $d = d', \chi = \chi'$ and f' is obtained from f by composition with a permutation of $\{1, 2, \dots, n/d\}$.

0.8. In the ungraded case, the objects in the block $\mathcal{D}_G(\mathfrak{g}^{nil})_{(L,C)}$ are obtained from C via parabolic induction (and decomposition) through any parabolic subgroup P of G containing L as a Levi subgroup. In the $\mathbf{Z}/m$-graded case, a first attempt to generalize parabolic induction would be to start with a parabolic subgroup of G compatible with the $\mathbf{Z}/m$-grading on $\mathfrak{g}$, as defined in the appendix of [L5]. However, such a parabolic induction does not produce all simple perverse sheaves in $\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_{\underline{\eta}}^{nil})$. Instead, we introduce a certain induction procedure which we call *spiral induction*; see Section 4. We introduce the notion of a spiral $\mathfrak{p}_*$ which is a sequence of subspaces $\mathfrak{p}_N \subset \mathfrak{g}_{\underline{N}}$, one for each $N \in \mathbf{Z}$, satisfying certain conditions; see Section 2. It turns out that spirals are the correct analogues of parabolic subalgebras in the $\mathbf{Z}/m$-graded case. Moreover, spiral induction includes the parabolic induction defined in the appendix of [L5] as special cases. In fact there are two kinds of spiral inductions, one giving objects in $\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_{\underline{\eta}}^{nil})$ and the other giving (assuming that $p > 0$) Fourier-Deligne transforms of objects in $\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_{-\underline{\eta}}^{nil})$. The latter may be viewed as an analogue of character sheaves in the $\mathbf{Z}/m$-graded setting.

0.9. We now discuss the contents of the various sections. Many arguments in this paper rely on results from [L4] concerning $\mathbf{Z}$-graded Lie algebras; in Section 1 we review some results from [L4] that we need. In Section 2 we introduce the ϵ-spirals attached to a $\mathbf{Z}/m$-graded Lie algebra and their splittings; the analogous concepts in the $\mathbf{Z}$-graded cases are the parabolic subalgebras compatible with the $\mathbf{Z}$-grading and their Levi subalgebras compatible with the $\mathbf{Z}$-grading. We also attach a canonical spiral to any element of $\mathfrak{g}_{\underline{\eta}}^{nil}$ which plays a crucial role in the arguments of this paper. In Section 3 we introduce the admissible systems, which eventually will be used to index the blocks in $\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_{\underline{\eta}}^{nil})$. In Section 4 we introduce the operation of spiral induction which is our main tool in the study of $\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_{\underline{\eta}}^{nil})$. In Sections 5 and 6 we calculate explicitly the Hom space between two spiral inductions, generalizing to the $\mathbf{Z}/m$-graded case a result from [L4]. This is used in Section 7 to prove Theorem 0.6. In Section 8 we introduce monomial and quasi-monomial complexes on $\mathfrak{g}_{\underline{\eta}}^{nil}$; we show that the monomial complexes (resp. quasi-monomial) complexes generate the appropriate Grothendieck group over $\mathbf{Q}(v)$ (resp. over $\mathbf{Z}[v, v^{-1}]$) where v is an indeterminate; this again generalizes to the $\mathbf{Z}/m$-graded case a result from [L4]. This result is of the same type as that which says that the plus part of a quantized enveloping algebra defined in terms of perverse sheaves is generated over $\mathbf{Q}(v)$ by monomials in the E_i and over $\mathbf{Z}[v, v^{-1}]$ by the products of divided powers of the E_i (which could be called quasi-monomials). In Section 9 we discuss the examples where $G = SL(V)$ or $G = Sp(V)$; in these cases we describe the spirals and in the case of $G = SL(V)$ we describe the blocks.

0.10. It is known that, in the ungraded case, each block of $\mathcal{D}_G(\mathfrak{g}^{nil})$ can be related to the category of representations of a certain Weyl group; if m is large, so that the $\mathbf{Z}/m$ grading of $\mathfrak{g}$ is a $\mathbf{Z}$-grading and $\mathfrak{g}_{\underline{\eta}}^{nil} = \mathfrak{g}_{\underline{\eta}}$, then each block of $\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_{\underline{\eta}}^{nil})$ is related to the category of representations of a certain graded affine Hecke algebra with possibly unequal parameters. In fact, without assumptions on m, each block of $\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_{\underline{\eta}}^{nil})$ is related to a certain graded double affine Hecke algebra (corresponding to an affine Weyl group attached to the block) with possibly unequal parameters; this will be considered in a sequel to this paper. We also plan to describe explicitly the blocks in the case where G is a classical group and relate them to cyclic quivers with extra structure. The case of the symplectic group is partially carried out in 9.5–9.7.

0.11. **Notation.** All algebraic varieties are assumed to be over $\mathbf{k}$; all algebraic groups are assumed to be affine. Let l be a prime number invertible in $\mathbf{k}$. For any algebraic variety X we denote by $\mathcal{D}(X)$ the bounded derived category of $\bar{\mathbf{Q}}_l$-complexes on X. Let $D : \mathcal{D}(X) \to \mathcal{D}(X)$ be Verdier duality.

For $K \in \mathcal{D}(X)$ we denote by $\mathcal{H}^n K$ the n-th cohomology sheaf of K and by $\mathcal{H}^n_x K$ the stalk of $\mathcal{H}^n K$ at $x \in X$.

If X' is a locally closed smooth irreducible subvariety of X with closure $\bar{X}'$ and $\mathcal{L}$ is an irreducible local system on X' we denote by $\mathcal{L}^\sharp \in \mathcal{D}(X)$ the intersection cohomolgy complex of $\bar{X}'$ with coefficients in $\mathcal{L}$, extended by 0 on $X - \bar{X}'$.

If X has a given action of an algebraic group H we denote by $\mathcal{D}_H(X)$ the corresponding equivariant derived category.

If H is an algebraic group we denote by H^0 the identity component of H, by $\mathcal{Z}_H$ the center of H. We set $\pi_0(H) = H/H^0$. Now assume that H is connected. We denote by $\mathfrak{L}H$ the Lie algebra of H and by U_H the unipotent radical of H. Let $\mathfrak{h} = \mathfrak{L}H$. If $\mathfrak{h}'$ is a Lie subalgebra of $\mathfrak{h}$ we write $e^{\mathfrak{h}'} \subset H$ for the closed connected subgroup of H such that $\mathfrak{L}(e^{\mathfrak{h}'}) = \mathfrak{h}'$, assuming that such a subgroup exists.

We shall often denote a collection $\{V_N; N \in \mathbf{Z}\}$ of vector spaces indexed by $N \in \mathbf{Z}$ by the symbol V_*.

If V is a $\mathbf{k}$-vector space, a $\mathbf{Z}$-grading on V is a collection of subspaces $V_* = \{V_k; k \in \mathbf{Z}\}$ such that $V = \oplus_{k \in \mathbf{Z}} V_k$; a $\mathbf{Z}/m$-grading on V is a collection of subspaces

$\{V_i; i \in \mathbf{Z}/m\}$ such that $V = \oplus_{i\in\mathbf{Z}/m} V_i$; a $\mathbf{Q}$-grading on V is a collection of subspaces $\{_\kappa V; \kappa \in \mathbf{Q}\}$ such that $V = \oplus_{\kappa\in\mathbf{Q}}(_\kappa V)$.

A $\mathbf{Z}$-grading for the Lie algebra $\mathfrak{h}$ is a $\mathbf{Z}$-grading $\mathfrak{h}_* = \{\mathfrak{h}_k; k \in \mathbf{Z}\}$ of $\mathfrak{h}$ as a vector space satisfying $[\mathfrak{h}_k, \mathfrak{h}_{k'}] \subset \mathfrak{h}_{k+k'}$ for all $k, k' \in \mathbf{Z}$; a $\mathbf{Z}/m$-grading for $\mathfrak{h}$ is a $\mathbf{Z}/m$-grading $\{\mathfrak{h}_i; i \in \mathbf{Z}/m\}$ of $\mathfrak{h}$ as a vector space satisfying $[\mathfrak{h}_i, \mathfrak{h}_{i'}] \subset \mathfrak{h}_{i+i'}$ for all $i, i' \in \mathbf{Z}/m$; a $\mathbf{Q}$-grading for $\mathfrak{h}$ is a $\mathbf{Q}$-grading $\{_\kappa\mathfrak{h}; \kappa \in \mathbf{Q}\}$ of $\mathfrak{h}$ as a vector space satisfying $[_\kappa\mathfrak{h}, {}_{\kappa'}\mathfrak{h}] \subset {}_{\kappa+\kappa'}\mathfrak{h}$ for all $\kappa, \kappa' \in \mathbf{Q}$.

Let Y_H be the set of homomorphisms of algebraic groups $\mathbf{k}^* \to H$. For $\lambda \in Y_H$ and $c \in \mathbf{Z}$, we define $c\lambda \in Y_H$ by $(c\lambda)(t) = \lambda(t^c)$ for $t \in \mathbf{k}^*$. We define an equivalence relation on $Y_H \times \mathbf{Z}_{>0}$ by $(\lambda, r) \sim (\lambda', r')$ whenever there exist c, c' in $\mathbf{Z}_{>0}$ such that $c\lambda = c'\lambda'$, $cr = c'r'$; the set of equivalence classes for this relation is denoted by $Y_{H,\mathbf{Q}}$. Let $\lambda/r = (1/r)\lambda$ be the equivalence class of (λ, r). Now $\lambda \mapsto \lambda/1$ identifies Y_H with a subset of $Y_{H,\mathbf{Q}}$. For $\kappa \in \mathbf{Q}, \mu \in Y_{H,\mathbf{Q}}$ we define $\kappa\mu \in Y_{H,\mathbf{Q}}$ by $\kappa\mu = (k\lambda)/(k'r)$, where $k \in \mathbf{Z}, k' \in \mathbf{Z}_{>0}, r \in \mathbf{Z}_{>0}, \lambda \in Y_H$ are such that $\kappa = k/k'$, $\mu = \lambda/r$; this is independent of the choices. In particular, we have $r\mu \in Y_H$ for some $r \in \mathbf{Z}_{>0}$.

Let $\lambda \in Y_H$. For $k \in \mathbf{Z}$ we set

$$^\lambda_k\mathfrak{h} = \{x \in \mathfrak{h}; \mathrm{Ad}(\lambda(t))x = t^k x \quad \forall t \in \mathbf{k}^*\}.$$

Note that $\{^\lambda_k\mathfrak{h}, k \in \mathbf{Z}\}$ is a $\mathbf{Z}$-grading of $\mathfrak{h}$.

Now let $\mu \in Y_{H,\mathbf{Q}}$. For $\kappa \in \mathbf{Q}$ we set $^\mu_\kappa\mathfrak{h} = {}^{r\mu}_{r\kappa}\mathfrak{h}$ where $r \in \mathbf{Z}_{>0}$ is chosen so that $r\mu \in Y_H$, $r\kappa \in \mathbf{Z}$. This is well defined (independent of the choice of r). Note that $\{^\mu_\kappa\mathfrak{h}, \kappa \in \mathbf{Q}\}$ is a $\mathbf{Q}$-grading of $\mathfrak{h}$.

0.12. Let H be a connected algebraic group acting on an algebraic variety X and let A, B be two H-equivariant semisimple complexes on X; let $j \in \mathbf{Z}$. We define a finite dimensional $\bar{\mathbf{Q}}_l$-vector space $\mathbf{D}_j(X, H; A, B)$ as in [L4, 1.7]. For the purpose of this paper, we can take the following formula as the definition of $\mathbf{D}_j(X, H; A, B)$:

(a) $\mathbf{D}_j(X, H; A, B) = \mathrm{Hom}_{\mathcal{D}_H(X)}(A, D(B)[-j])^*$.

Let $d_j(X; A, B) = \dim \mathbf{D}_j(X, H; A, B)$, $\{A, B\} = \sum_{j\in\mathbf{Z}} d_j(X; A, B)v^{-j} \in \mathbf{N}((v))$ where v is an indeterminate.

If A, B are H-equivariant simple perverse sheaves on X, then

$$\{A, B\} \in 1 + v\mathbf{N}[[v]] \text{ if } B \cong D(A),$$
$$\{A, B\} \in v\mathbf{N}[[v]] \text{ if } B \ncong D(A).$$

(See [L4, 1.8(d)].)

For an algebraic variety X we denote by ρ_X the map $X \to$ (point).

Let v be an indeterminate and let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$. Let $^-\colon \mathbf{Q}(v) \to \mathbf{Q}(v)$ be the field involution such that $\bar{v} = v^{-1}$. This restricts to a ring involution $^-\colon \mathcal{A} \to \mathcal{A}$.

For any $\eta \in \mathbf{Z} - \{0\}$ we define $\dot{\eta} = \eta/|\eta| \in \{1, -1\}$ where $|\eta|$ is the absolute value of η.

1. Recollections on Z-graded Lie algebras

In this section we recall notation and results from [L4] that will be used in this paper.

1.1. In this section we fix a connected reductive group H; let $\mathfrak{h} = \mathfrak{L}H$.

Let J^H be the variety consisting of all triples $(e, h, f) \in \mathfrak{h}^3$ such that $[h, e] = 2e, [h, f] = -2f, [e, f] = h$ (then e, f are necessarily in $\mathfrak{h}^{nil}$). If $\phi = (e, h, f) \in J^H$, there is a unique homomorphism of algebraic groups $\tilde{\phi} : SL_2(\mathbf{k}) \to H$ such that the differential of $\tilde{\phi}$ carries $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ to e, h, f respectively; we then define $\iota_\phi \in Y_H$ by $\iota_\phi(t) = \tilde{\phi}\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$.

1.2. In the remainder of this section we assume that a $\mathbf{Z}$-grading $\mathfrak{h}_*$ for $\mathfrak{h}$ is given. Then there exists $\lambda \in Y_H$ and $r \in \mathbf{Z}_{>0}$ with $\mathfrak{h}_k = {}^{\lambda}_{rk}\mathfrak{h}$ for all $k \in \mathbf{Z}$. (It follows that ${}^{\lambda}_{\kappa}\mathfrak{h} = 0$ for all $\kappa \in \mathbf{Q} - r\mathbf{Z}$.)

(In this paper we will often refer to results in [L4], even though, strictly speaking, in [L4] a stronger assumption on the $\mathbf{Z}$-grading of $\mathfrak{h}$ is made, namely that r above can be taken to be 1. Note that the results of [L4] hold with the same proof when the stronger assumption is replaced by the present assumption.)

We have $\mathfrak{h}_k \subset \mathfrak{h}^{nil}$ for any $k \in \mathbf{Z} - \{0\}$. Note that $\mathfrak{h}_0$ is a Lie subalgebra of $\mathfrak{h}$ and that $H_0 := e^{\mathfrak{h}_0} \subset H$ is well defined and it acts by the Ad-action on each $\mathfrak{h}_k$. If $k \neq 0$, this action has only finitely many orbits (see [L4, 3.5]); we denote by $\overset{\circ}{\mathfrak{h}}_k$ the unique open H_0-orbit in $\mathfrak{h}_k$.

Let $\eta \in \mathbf{Z} - \{0\}$.

(a) *We say that the $\mathbf{Z}$-grading $\mathfrak{h}_*$ of $\mathfrak{h}$ is η-rigid if there exists $\iota \in Y_H$ such that* (i), (ii) *below hold.*

(i) ${}^{\iota}_{k}\mathfrak{h} = \mathfrak{h}_{\eta k/2}$ *for any $k \in \mathbf{Z}$ such that $\eta k/2 \in \mathbf{Z}$ and ${}^{\iota}_{k}\mathfrak{h} = 0$ for any $k \in \mathbf{Z}$ such that $\eta k/2 \notin \mathbf{Z}$;*

(ii) $\iota = \iota_\phi$ *for some $\phi = (e, h, f) \in J^H$ such that $e \in \overset{\circ}{\mathfrak{h}}_\eta$, $h \in \mathfrak{h}_0$, $f \in \mathfrak{h}_{-\eta}$.* It follows that $2k' \in \eta\mathbf{Z}$ whenever $\mathfrak{h}_{k'} \neq 0$. Note that ι is unique if it exists, since, by (ii), $\iota(\mathbf{k}^*)$ is contained in the derived group of H.

We show:

(b) *In the setup of* (a), *let $\phi' = (e', h', f') \in J^H$ be such that $e' \in \overset{\circ}{\mathfrak{h}}_\eta$, $h' \in \mathfrak{h}_0$, $f' \in \mathfrak{h}_{-\eta}$. Let $\iota' = \iota_{\phi'}$. Then $\iota' = \iota$.*

Let ϕ be as in (ii). Using [L4, 3.3], we can find $g_0 \in H_0$ such that $\mathrm{Ad}(g_0)$ carries ϕ to ϕ'. It follows that $\mathrm{Ad}(g_0)\iota(t) = \iota'(t)$ for any $t \in \mathbf{k}^*$. For $k \in \mathbf{Z}$ such that $\eta k/2 \in \mathbf{Z}$ we have

$$ {}^{\iota'}_{k}\mathfrak{h} = \mathrm{Ad}(g_0)({}^{\iota}_{k}\mathfrak{h}) = \mathrm{Ad}(g_0)\mathfrak{h}_k = \mathfrak{h}_k; $$

for $k \in \mathbf{Z}$ such that $\eta k/2 \notin \mathbf{Z}$ we have

$$ {}^{\iota'}_{k}\mathfrak{h} = \mathrm{Ad}(g_0)({}^{\iota}_{k}\mathfrak{h}) = 0, $$

$$ {}^{\iota'}_{2k\eta}\mathfrak{h} = \mathrm{Ad}(g_0)({}^{\iota}_{2k\eta}\mathfrak{h}) = \mathrm{Ad}(g_0)\mathfrak{h}_k = \mathfrak{h}_k. $$

Thus ι' satisfies the defining properties of ι in (a). By uniqueness we have $\iota' = \iota$ as required.

Let $\mathcal{I}(\mathfrak{h}_\eta)$ be the set of all pairs $(\mathcal{O}, \mathcal{L})$ where $\mathcal{O}$ is an H_0-orbit in $\mathfrak{h}_\eta$ and $\mathcal{L}$ is an H_0-equivariant irreducible local system on $\mathfrak{h}_\eta$ (up to isomorphism).

Let $\mathcal{Q}(\mathfrak{h}_\eta)$ be the category of $\bar{\mathbf{Q}}_l$-complexes on $\mathfrak{h}_\eta$ which are direct sums of shifts of simple H_0-equivariant perverse sheaves on $\mathfrak{h}_\eta$. There are up to isomorphism only finitely many such simple perverse sheaves; they form a set in bijection with $\mathcal{I}(\mathfrak{h}_\eta)$.

An H_0-equivariant perverse sheaf A on $\mathfrak{h}_\eta$ is said to be *cuspidal* if there exists a nilpotent H-orbit $\mathcal{C}$ in $\mathfrak{h}$ and an irreducible H-equivariant cuspidal local system $\mathcal{F}$

on $\mathcal{C}$ such that $\overset{\circ}{\mathfrak{h}}_\eta \subset \mathcal{C}$ and $A|_{\overset{\circ}{\mathfrak{h}}_\eta} = \mathcal{F}|_{\overset{\circ}{\mathfrak{h}}_\eta}[\dim \mathfrak{h}_\eta]$. If such $(\mathcal{C}, \mathcal{F})$ exists, it is unique; see [L4, 4.2(c)]. Note that if A is cuspidal, then it is necessarily a simple perverse sheaf.

(c) *If there exists a cuspidal H_0-equivariant perverse sheaf A on $\mathfrak{h}_\eta$, the grading $\mathfrak{h}_*$ of $\mathfrak{h}$ is necessarily η-rigid; moreover, we have $A|_{\mathfrak{h}_\eta - \overset{\circ}{\mathfrak{h}}_\eta} = 0$.*

(See [L4, 4.4(a), 4.4(b)].)

In the setup of (c), the element $\iota \in Y_H$ provided by (a) is known to satisfy

(d) ${}^\iota_k\mathfrak{h} = 0$ unless $k \in 2\mathbf{Z}$;

we deduce that:

(e) *If $k' \in \mathbf{Z}$ and $\mathfrak{h}_{k'} \neq 0$, then $k'/\eta \in \mathbf{Z}$.*

1.3. **Parabolic induction.** In the setup of 1.2 assume that P is a parabolic subgroup of H with $\mathfrak{p} := \mathfrak{L}P$ satisfying $\mathfrak{p} = \oplus_{k\in\mathbf{Z}}\mathfrak{p}_k$ where $\mathfrak{p}_k = \mathfrak{p} \cap \mathfrak{h}_k$. We set $U = U_P, L = P/U, \mathfrak{u} = \mathfrak{L}U, \mathfrak{l} = \mathfrak{L}L = \mathfrak{p}/\mathfrak{u}$. We have $\mathfrak{u} = \oplus_{k\in\mathbf{Z}}\mathfrak{u}_k$ where $\mathfrak{u}_k = \mathfrak{u}\cap\mathfrak{h}_k$. Setting $\mathfrak{l}_k = \mathfrak{p}_k/\mathfrak{u}_k$, we have $\mathfrak{l} = \oplus_{k\in\mathbf{Z}}\mathfrak{l}_k$; this gives a $\mathbf{Z}$-grading of the Lie algebra $\mathfrak{l}$.

Now $\mathfrak{p}_0$ is a parabolic subalgebra of the reductive Lie algebra $\mathfrak{h}_0$; we have $\mathfrak{p}_0 = \mathfrak{L}P_0$ where P_0 is a parabolic subgroup of the connected reductive group H_0. Let L_0 be the image of P_0 under the obvious homomorphism $P \to L$. Then $L_0 = e^{\mathfrak{l}_0} \subset L$. Now P_0 acts by the Ad-action on each $\mathfrak{p}_k$. Let $\pi : \mathfrak{p}_\eta \to \mathfrak{l}_\eta$ be the obvious projection. We have a diagram

$$\mathfrak{l}_\eta \xleftarrow{a} H_0 \times \mathfrak{p}_\eta \xrightarrow{b} E \xrightarrow{c} \mathfrak{h}_\eta,$$

where

$$E = \{(hP_0, z) \in H_0/P_0 \times \mathfrak{h}_\eta; \mathrm{Ad}(h^{-1})z \in \mathfrak{p}_\eta\},$$

$$a(h,z) = \pi(\mathrm{Ad}(h^{-1})z), b(h,z) = (hP_0, z), c(gP_0, z) = z.$$

Now a is smooth with connected fibers, b is a principal P_0-bundle and c is proper. If $A \in \mathcal{Q}(\mathfrak{l}_\eta)$, then a^*A is a P_0-equivariant semisimple complex on $H_0 \times \mathfrak{p}_\eta$ hence there is a well-defined semisimple complex A_1 on E such that $b^*A_1 = a^*A$. Since c is proper, the complex

$$\mathrm{ind}_{\mathfrak{p}_\eta}^{\mathfrak{h}_\eta}(A) := c_! A_1$$

belongs to $\mathcal{Q}(\mathfrak{h}_\eta)$. For $B \in \mathcal{D}(\mathfrak{h}_\eta)$ we can form

$$\mathrm{res}_{\mathfrak{p}_\eta}^{\mathfrak{h}_\eta}(B) := \pi_!(B|_{\mathfrak{p}_\eta}) \in \mathcal{D}(\mathfrak{l}_\eta).$$

Thus we have functors $\mathrm{res}_{\mathfrak{p}_\eta}^{\mathfrak{h}_\eta} : \mathcal{D}(\mathfrak{h}_\eta) \to \mathcal{D}(\mathfrak{l}_\eta)$, $\mathrm{ind}_{\mathfrak{p}_\eta}^{\mathfrak{h}_\eta} : \mathcal{Q}(\mathfrak{l}_\eta) \to \mathcal{Q}(\mathfrak{h}_\eta)$.

When $\tilde{\mathfrak{l}}$ is a Levi subalgebra of $\mathfrak{p}$ such that $\tilde{\mathfrak{l}} = \oplus_{k\in\mathbf{Z}}\tilde{\mathfrak{l}}_k$ with $\tilde{\mathfrak{l}}_k = \tilde{\mathfrak{l}} \cap \mathfrak{h}_k$, we will sometime consider $\mathrm{ind}_{\mathfrak{p}_\eta}^{\mathfrak{h}_\eta}(A)$ with $A \in \mathcal{Q}(\tilde{\mathfrak{l}}_\eta)$ by identifying $\tilde{\mathfrak{l}}_\eta = \mathfrak{l}_\eta$ in an obvious way and A with an object in $\mathcal{Q}(\mathfrak{l}_\eta)$.

1.4. In the setup of 1.3 let S'_P be the set of Levi subgroups of P and let S_P be the set of all $M \in S'_P$ such that, setting $\mathfrak{L}M = \mathfrak{m}$, $\mathfrak{m}_k = \mathfrak{m} \cap \mathfrak{h}_k$, we have $\mathfrak{m} = \oplus_{k\in\mathbf{Z}}\mathfrak{m}_k$, or equivalently such that $\mathrm{Ad}(\lambda(t))\mathfrak{m} = \mathfrak{m}$ for all $t \in \mathbf{k}^*$. We have $S_P \neq \emptyset$; indeed, we can find $M \in S'_P$ such that $\lambda(k^*) \subset M$; then $M \in S_P$. Since U acts simply transitively by conjugation on S'_P, it follows that:

(a) *The unipotent group $\{u \in U; u\lambda(t) = \lambda(t)u \quad \forall t \in \mathbf{k}^*\}$ acts simply transitively by conjugation on S_P.*

1.5. **Blocks for $\mathcal{Q}(\mathfrak{h}_\eta)$.** Let $\mathfrak{M}_\eta(H)$ be the set of all systems

$$(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}),$$

where M is a Levi subgroup of some parabolic subgroup of H, $\mathfrak{m} = \mathfrak{L}M$, $\mathfrak{m}_*$ is a $\mathbf{Z}$-grading of $\mathfrak{m}$ such that $\mathfrak{m}_k = \mathfrak{m} \cap \mathfrak{h}_k$ for all k, $M_0 = e^{\mathfrak{m}_0} \subset M$ and $\tilde{C}$ is a cuspidal M_0-equivariant perverse sheaf on $\mathfrak{m}_\eta$ (up to isomorphism). Note that H_0 acts by conjugation on $\mathfrak{M}_\eta(H)$. Let $\underline{\mathfrak{M}}_\eta(H)$ be the set of orbits for this action.

In the setup of 1.2 assume that A is a simple H_0-equivariant perverse sheaf on $\mathfrak{h}_\eta$. By [L4, 7.5]:

(a) *There exists $P, L, L_0, \mathfrak{p}, \mathfrak{l}$ as in 1.3 and a cuspidal L_0-equivariant perverse sheaf C on $\mathfrak{l}_\eta$ such that some shift of A is a direct summand of* $\operatorname{ind}_{\mathfrak{p}_\eta}^{\mathfrak{h}_\eta}(C)$.

Assume now that $P', L', L'_0, \mathfrak{p}', \mathfrak{l}'$ is another quintuple like $P, L, L_0, \mathfrak{p}, \mathfrak{l}$ and that C' is a cuspidal L'_0-equivariant perverse sheaf on $\mathfrak{l}'_\eta$ such that some shift of A is a direct summand of $\operatorname{ind}_{\mathfrak{p}'_\eta}^{\mathfrak{h}_\eta}(C')$.

Let $M \in S_P, M' \in S_{P'}$, let $\mathfrak{L}M = \mathfrak{m} = \oplus_k \mathfrak{m}_k$ be as in 1.4 and let $\mathfrak{L}M' = \mathfrak{m}' = \oplus_k \mathfrak{m}'_k$ where $\mathfrak{m}'_k = \mathfrak{m}' \cap \mathfrak{h}_k$. Let $M_0 = e^{\mathfrak{m}_0} \subset M$, $M'_0 = e^{\mathfrak{m}'_0} \subset M'$. We can identify $M, M_0, \mathfrak{m}, \mathfrak{m}_k$ with $L, L_0, \mathfrak{l}, \mathfrak{l}_k$ via $P \to L$ and we can identify $M', M'_0, \mathfrak{m}', \mathfrak{m}'_k$ with $L', L'_0, \mathfrak{l}', \mathfrak{l}'_k$ via $P' \to L'$. Then C (resp. C') becomes a cuspidal M_0-equivariant (resp. M'_0-equivariant) perverse sheaf $\tilde{C}$ (resp. $\tilde{C}'$) on $\mathfrak{m}_\eta$ (resp. $\mathfrak{m}'_\eta$).

Using the last sentence of [L4, 15.3], we see that there exists $h \in H_0$ such that $\operatorname{Ad}(h)$ carries $M, M_0, \mathfrak{m}, \mathfrak{m}_k$ to $M', M'_0, \mathfrak{m}', \mathfrak{m}'_k$ and $\tilde{C}$ to $\tilde{C}'$. Thus, we have:

(b) *$A \mapsto (M, M_0, \mathfrak{m}, \mathfrak{m}_k, \tilde{C})$ is a well-defined map from the set of (isomorphism classes) of simple H_0-equivariant perverse sheaves on $\mathfrak{h}_\eta$ to the set $\underline{\mathfrak{M}}_\eta(H)$.*

1.6. Let $(M, M_0, \mathfrak{m}, \mathfrak{m}_k, \tilde{C}) \in \mathfrak{M}_\eta(H)$. We show:

(a) *There exists a parabolic subgroup P of H such that M is a Levi subgroup of P and such that, setting $\mathfrak{p} = \mathfrak{L}P$, $\mathfrak{p}_k = \mathfrak{p} \cap \mathfrak{h}_k$, we have $\mathfrak{p} = \oplus_{k \in \mathbf{Z}} \mathfrak{p}_k$.*

Let $\mathcal{Z} = \mathcal{Z}_M^0$. Then $\mathfrak{z} = \mathfrak{L}\mathcal{Z}$ is the center of $\mathfrak{m}$. Since $\mathfrak{m}_0$ is a Levi subalgebra of a parabolic subalgebra of $\mathfrak{m}$, we have $\mathfrak{z} \subset \mathfrak{m}_0$ hence $\mathcal{Z} \subset M_0$. We can find $\lambda_1 \in Y_{\mathcal{Z}}$ such that the centralizer of $\lambda_1(\mathbf{k}^*)$ in H is equal to the centralizer of $\mathcal{Z}$ in H which equals M. Let $\lambda \in Y_H, r$ be as in 1.2. Then $\lambda(\mathbf{k}^*) \subset \mathcal{Z}_{H_0}$. Now $\lambda_1(\mathbf{k}^*) \subset \mathcal{Z}$ hence $\lambda_1(\mathbf{k}^*) \subset H_0$. It follows that $\lambda_1(t)\lambda(t') = \lambda(t')\lambda_1(t)$ for any t, t' in $\mathbf{k}^*$. Thus we have $\mathfrak{h} = \oplus_{k \in \mathbf{Z}, k' \in \mathbf{Z}} ({}^{\lambda}_{kr}\mathfrak{h} \cap {}^{\lambda_1}_{k'}\mathfrak{h})$. Since the centralizer of $\lambda_1(\mathbf{k}^*)$ in $\mathfrak{h}$ equals $\mathfrak{m}$, we have $\mathfrak{m} = \oplus_{k \in \mathbf{Z}} ({}^{\lambda}_{kr}\mathfrak{h} \cap {}^{\lambda_1}_{0}\mathfrak{h})$. We set

$$\mathfrak{p} = \oplus_{k \in \mathbf{Z}, k' \in \mathbf{Z}_{\geq 0}} ({}^{\lambda}_{kr}\mathfrak{h} \cap {}^{\lambda_1}_{k'}\mathfrak{h}).$$

Clearly, $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{h}$ with Levi subalgebra $\mathfrak{m}$ and such that, setting $\mathfrak{p}_k = \mathfrak{p} \cap \mathfrak{h}_k$, we have $\mathfrak{p} = \oplus_{k \in \mathbf{Z}} \mathfrak{p}_k$. This proves (a).

1.7. To any $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{M}_\eta(H)$ we associate a simple perverse sheaf A in $\mathcal{Q}(\mathfrak{h}_\eta)$ as follows. Let $\mathcal{O}$ be the H_0-orbit in $\mathfrak{h}_\eta$ which contains $\mathring{\mathfrak{m}}_\eta$. Let $\mathcal{L}'$ be the irreducible M_0-equivariant local system on $\mathring{\mathfrak{m}}_\eta$ such that $\tilde{C}|_{\mathring{\mathfrak{m}}_\eta} = \mathcal{L}'[\dim \mathfrak{m}_\eta]$. By [L4, 11.2], there is a well-defined irreducible H_0-equivariant local system $\mathcal{L}$ on $\mathcal{O}$ such that $\mathcal{L}|_{\mathring{\mathfrak{m}}_\eta} = \mathcal{L}'$. By definition, A is the simple perverse sheaf on $\mathfrak{h}_\eta$ such that $\operatorname{supp} A$ is contained in the closure of $\mathcal{O}$ and $A|_{\mathcal{O}} = \mathcal{L}[\dim \mathcal{O}]$.

1.8. Assume that the $\mathbf{Z}$-grading $\mathfrak{h}_*$ of $\mathfrak{h}$ is η-rigid. A perverse sheaf A in $\mathcal{Q}(\mathfrak{h}_\eta)$ is said to be *η-semicuspidal* if $\operatorname{supp} A = \mathfrak{h}_\eta$ and A is attached to some

$$(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{M}_\eta(H),$$

as in 1.7 (in particular, A is a simple perverse sheaf). In this case we have $\overset{\circ}{\mathfrak{m}}_\eta \subset \overset{\circ}{\mathfrak{h}}_\eta$; moreover,

(a) *H_0 acts transitively on the set of systems $(M, M_0, \mathfrak{p}, \mathfrak{p}_*, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$ such that $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{M}_\eta(H)$, A is attached to $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$ as in 1.7, $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{h}$ with Levi subalgebra $\mathfrak{m}$ and $\mathfrak{p} = \oplus_{k\in\mathbf{Z}}\mathfrak{p}_k$ where $\mathfrak{p}_k = \mathfrak{p} \cap \mathfrak{h}_k$.* (See [L4, 11.9].)

If $(M, M_0, \mathfrak{p}, \mathfrak{p}_*, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$ is as in (a), then

(b) $$\operatorname{ind}_{\mathfrak{p}_\eta}^{\mathfrak{h}_\eta}(\tilde{C}) \cong \oplus_j A[-2s_j][\dim \mathfrak{m}_\eta - \dim \mathfrak{h}_\eta],$$

where $s_j \in \mathbf{N}$ are defined as follows. Choose $\phi = (e, h, f) \in J^H$ as in 1.2(ii); let $H_\phi = \{g \in H; \operatorname{Ad}(g)(e) = e, \operatorname{Ad}(g)(h) = h, \operatorname{Ad}(g)(f) = f\}$, let $\mathcal{B}$ be the variety of Borel subgroups of H_ϕ^0; then s_j are defined by $\rho_{\mathcal{B}!}\bar{\mathbf{Q}}_l = \oplus_j \bar{\mathbf{Q}}_l[-2s_j]$. (See [L4, 11.13].)

1.9. Let $\mathcal{X}$ be the set of all systems $(M, M_0, \mathfrak{p}, \mathfrak{p}_*, \mathfrak{m}, \mathfrak{m}_*, \tilde{A})$ where $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{h}$ with Levi subalgebra $\mathfrak{m}$, $\mathfrak{p} = \oplus_{k\in\mathbf{Z}}\mathfrak{p}_k$, $\mathfrak{m} = \oplus_{k\in\mathbf{Z}}\mathfrak{m}_k$ where $\mathfrak{p}_k = \mathfrak{p} \cap \mathfrak{h}_k$, $\mathfrak{m}_k = \mathfrak{m} \cap \mathfrak{h}_k$, $M = e^{\mathfrak{m}}, M_0 = e^{\mathfrak{m}_0}$ and $\tilde{A}$ is a simple perverse sheaf in $\mathcal{Q}(\mathfrak{m}_\eta)$ (up to isomorphism) which is η-semicuspidal. We have the following result; see [L4, 13.3].

(a) *Let $A_1 \in \mathcal{Q}(\mathfrak{h}_\eta)$. There exists $C_1, C_2, \ldots, C_t, C_{t+1}, \ldots, C_{t+t'}$ in $\mathcal{Q}(\mathfrak{h}_\eta)$ such that*

$$A_1 \oplus C_1 \oplus C_2 \oplus \ldots \oplus C_t = C_{t+1} \oplus \ldots \oplus C_{t+t'}$$

and each C_j is of the form $\operatorname{ind}_{\mathfrak{p}_\eta}^{\mathfrak{h}_\eta}(\tilde{A})[a_j]$ for some $(M, M_0, \mathfrak{p}, \mathfrak{p}_*, \mathfrak{m}, \mathfrak{m}_*, \tilde{A}) \in \mathcal{X}$ (depending on j) and some $a_j \in \mathbf{Z}$.

Erratum to [L4]. In the definition of a good object in the second paragraph of [L4, 13.2], one should insert the words "shifts of" after "direct sum of" (twice).

1.10. Let $s \in \mathbf{Z} - \{0\}$. We show:

(a) *the subspace $\mathfrak{h}^{(1)} := \oplus_{k\in s\mathbf{Z}}\mathfrak{h}_k$ of $\mathfrak{h}$ is the Lie algebra of a well-defined connected reductive subgroup $H^{(1)}$ of H.*

We can assume that $s > 0$. We shall define $e \in Z_{\geq 0}$ as follows: if $p = 0$ we have $e = 0$; if $p > 0$ we define e by $s = s'p^e$, where $s' \in \mathbf{Z}_{>0}$ is not divisible by p. We shall argue by induction on e. (When $p = 0$ we only have to consider the case $e = 0$.) Assume first that $e = 0$.

Let $\bar{H}$ be the adjoint group of H and let $\bar{\mathfrak{h}}$ be its Lie algebra. Then $\bar{\mathfrak{h}}$ inherits a $\mathbf{Z}$-grading $\bar{\mathfrak{h}} = \oplus_k\bar{\mathfrak{h}}_k$ from $\mathfrak{h}$. If we assume known that $\bar{\mathfrak{h}}^{(1)} := \oplus_{k\in s\mathbf{Z}}\bar{\mathfrak{h}}_k$ is the Lie algebra of a well-defined connected reductive subgroup $\bar{H}^{(1)}$ of $\bar{H}$, then we can take $H^{(1)}$ to be the identity component of the inverse image of $\bar{H}^{(1)}$ under the obvious map $H \to \bar{H}$. Thus we can assume that H is adjoint. Let $\lambda \in Y_H$ be such that ${}^{\lambda}_{k}\mathfrak{h} = \mathfrak{h}_k$ for all k. Let ζ' be a primitive s-th root of 1 in $\mathbf{k}$. (Note that if $p > 0$, $s = s'$ is not divisible by p.) We define $\omega : H \to H$ by $\omega(g) = \operatorname{Ad}(\lambda(\zeta'))(g)$; this is an automorphism of H. The automorphism $\omega' : \mathfrak{h} \to \mathfrak{h}$ induced by ω sends $x \in \mathfrak{h}_k$ (where $k \in \mathbf{Z}$) to $\zeta'^k x$. Hence $\omega^s = 1$ and $\mathfrak{h}^{(1)}$ is equal to $\{x \in \mathfrak{h}; \omega(x) = x\}$. Let $H^{(1)}$ be the identity component of $\{g \in H; \omega(g) = g\}$. This is a connected reductive group with Lie algebra $\mathfrak{h}^{(1)}$. Thus (a) is proved in the case $e = 0$. We now assume that $e \geq 1$ hence $p > 0$. We can find an element $x_0 \in \mathfrak{h}$ such that $[x_0, x] = kx$ for any $k \in \mathbf{Z}$ and any $x \in \mathfrak{g}_k$. (We can take x_0 in the image of the tangent map of $\lambda : \mathbf{k}^* \to H$.) Let $\tilde{\mathfrak{h}} = \{x \in \mathfrak{h}; [x_0, x] = 0\}$. We have $\tilde{\mathfrak{h}} = \oplus_{k \in p\mathbf{Z}} \mathfrak{h}_k$. Let $\tilde{H}$ be the identity component of $\{g \in H; \mathrm{Ad}(g)x_0 = x_0\}$. Since $x_0 \in \mathfrak{h}$ is semisimple, it follows that $\tilde{H}$ is reductive with Lie algebra $\tilde{\mathfrak{h}}$. We define a $\mathbf{Z}$-grading $\tilde{\mathfrak{h}} = \oplus_{k' \in \mathbf{Z}} \tilde{\mathfrak{h}}_{k'}$ by $\tilde{\mathfrak{h}}_{k'} = \mathfrak{h}_{pk'}$. By the induction hypothesis applied to $\tilde{H}, \tilde{\mathfrak{h}}$ we see that there is a well-defined connected reductive subgroup $\tilde{H}^{(1)}$ of $\tilde{H}$ whose Lie algebra is $\oplus_{k' \in (s/p)\mathbf{Z}} \tilde{\mathfrak{h}}_{k'} = \oplus_{k' \in (s/p)\mathbf{Z}} \mathfrak{h}_{pk'} = \oplus_{k \in s\mathbf{Z}} \mathfrak{h}_k = \mathfrak{h}^{(1)}$. We can take $H^{(1)} = \tilde{H}^{(1)}$. This completes the inductive proof.

2. $\mathbf{Z} \mapsto$-GRADINGS AND ϵ-SPIRALS

In this section we introduce the key notion of this paper, namely a spiral. Spirals are analogues in the $\mathbf{Z}/m$-graded setting of parabolic subalgebras in the ungraded or $\mathbf{Z}$-graded setting. We also attach a canonical spiral to each nilpotent element in $\mathfrak{g}_\delta$.

2.1. In the rest of this paper, $m \geq 1$, G, $\mathfrak{g} = \oplus_{i \in \mathbf{Z}/m} \mathfrak{g}_i$ are as in 0.1 and ζ, ϑ, θ are as in 0.5. Recall that for $i \in \mathbf{Z}/m$ we have $\mathfrak{g}_i = \{x \in \mathfrak{g}; \theta(x) = \zeta^i x\}$ and that $\vartheta : G \to G$ is the (semisimple) automorphism of G which induces $\theta : \mathfrak{g} \to \mathfrak{g}$; note that $\theta(\mathrm{Ad}(g)x) = \mathrm{Ad}(\vartheta(g))\theta(x)$ for all $x \in \mathfrak{g}$, $g \in G$.

We shall fix $\delta \in \mathbf{Z}/m$.

For any semisimple automorphism $\gamma : G \to G$, we set $G^\gamma = \{g \in G; \gamma(g) = g\}$. By a theorem of Steinberg [St],

(a) *G^γ is a connected reductive subgroup of G.*

Now $\mathfrak{g}_{\underline{0}}$ is a Lie subalgebra of $\mathfrak{g}$. Recall that $G_{\underline{0}} = G^\vartheta$ and that the Ad-action of $G_{\underline{0}}$ on $\mathfrak{g}$ leaves stable $\mathfrak{g}_i$ and its closed subset $\mathfrak{g}_i^{nil} := \mathfrak{g}_i \cap \mathfrak{g}^{nil}$ for any $i \in \mathbf{Z}/m$.

Let $\mathfrak{G}$ be the set of subgroups of G of the form $G^{\mathrm{Ad}(\tau)\vartheta}$ for some semisimple element of finite order $\tau \in G_{\underline{0}}$; by (a), any group in $\mathfrak{G}$ is a connected reductive subgroup of G. For example, we have $G_{\underline{0}} \in \mathfrak{G}$; hence we have $G_{\underline{0}} = e^{\mathfrak{g}_{\underline{0}}}$.

2.2. Let $\langle , \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbf{k}$ be a Killing form; it is nondegenerate and it satisfies $\langle \mathfrak{g}_i, \mathfrak{g}_j \rangle = 0$ whenever $i + j \neq \underline{0}$ in $\mathbf{Z}/m$. Hence for any $i \in \mathbf{Z}/m$, $\langle , \rangle : \mathfrak{g}_i \times \mathfrak{g}_{-i} \to \mathbf{k}$ is nondegenerate.

2.3. **The Morozov-Jacobson theorem in the $\mathbf{Z}/m$-graded setting.** We set $J = J^G$; see 1.1. For $x \in \mathfrak{g}^{nil}$ let $J(x) = \{(e, h, f) \in J; e = x\}$, $G(x) = \{g \in G; \mathrm{Ad}(g)x = x\}$ and let $U = U_{G(x)^0}$. Recall the following result of Morozov-Jacobson and Kostant; see [Ko].

(a) *We have $J(x) \neq \emptyset$. The U-action on $J(x)$ given by*

$$u : (e, h, f) \mapsto u(e, h, f) := (e, \mathrm{Ad}(u)h, \mathrm{Ad}(u)f)$$

is simply transitive.

Assume now that $x \in \mathfrak{g}_\delta^{nil}$. We set

$$J_\delta(x) = \{(e, h, f) \in J(x); e = x, h \in \mathfrak{g}_{\underline{0}}, f \in \mathfrak{g}_{-\delta}\}.$$

We show:

(b) *We have $J_\delta(x) \neq \emptyset$. The $(U \cap G_{\underline{0}})$-action on $J_\delta(x)$ (restriction of the U-action in* (a)*) is simply transitive.*

If $(e,h,f)\in J(x)$, then $(\zeta^{-\delta}e,h,\zeta^{\delta}f)\in J_\delta(\zeta^{-\delta}x)$ and

$$(\zeta^{-\delta}\theta(e),\theta(h),\zeta^{\delta}\theta(f))\in J(\zeta^{-\delta}\theta(x))=J(x)$$

(we use that $\theta(e)=\zeta^\delta e$). Hence $(e,h,f)\mapsto(\zeta^{-\delta}\theta(e),\theta(h),\zeta^{\delta}\theta(f))$ is a morphism $\theta':J(x)\to J(x)$. Next we note that $g\mapsto\vartheta(g)$ defines a homomorphism $G(x)\to G(x)$. (If $\mathrm{Ad}(g)x=x$, then $\theta(x)=\theta(\mathrm{Ad}(g)x)=\mathrm{Ad}(\vartheta(g))\theta(x)$. Since $\theta(x)=\zeta^\delta x$, we see that $\zeta^\delta x=\mathrm{Ad}(\vartheta(g))\zeta^\delta x$ hence $x=\mathrm{Ad}(\vartheta(g))x$ and $\vartheta(g)\in G(x)$.) This restricts to a homomorphism $\theta'':U\to U$ with fixed point set $U^{\theta''}$. For $u\in U$, $(e,h,f)\in J(x)$ we have $\theta'(u(e,h,f))=\theta''(u)\theta'(e,h,f)$. By (a), $J(x)$ is an affine space. Since $\theta'^m=1$ and m is invertible in $\mathbf{k}$, the fixed point set $J(x)^{\theta'}$ is nonempty. Since the U-action on $J(x)$ is simply transitive, it follows that this restricts to a simply transitive action of $U^{\theta''}$ on $J(x)^{\theta'}$. We have $J(x)^{\theta'}=J_\delta(x)$ and $U^{\theta''}=U\cap G_{\underline{0}}$. We see that (b) holds.

2.4. Let $\lambda\in Y_{G_{\underline{0}}}$ (resp. $\mu\in Y_{G_{\underline{0}},\mathbf{Q}}$). Since λ (resp. μ) can be viewed as an element of Y_G (resp. $Y_{G,\mathbf{Q}}$), the decomposition $\mathfrak{g}=\oplus_{k\in\mathbf{Z}}({}^\lambda_k\mathfrak{g})$ (resp. $\mathfrak{g}=\oplus_{\kappa\in\mathbf{Q}}({}^\mu_\kappa\mathfrak{g})$) is defined as in 1.1. For $i\in\mathbf{Z}/m$ and for $k\in\mathbf{Z}$ (resp. $\kappa\in\mathbf{Q}$) we set ${}^\lambda_k\mathfrak{g}_i={}^\lambda_k\mathfrak{g}\cap\mathfrak{g}_i$ (resp. ${}^\mu_\kappa\mathfrak{g}_i={}^\mu_\kappa\mathfrak{g}\cap\mathfrak{g}_i$); we then have $\mathfrak{g}_i=\oplus_{k\in\mathbf{Z}}({}^\lambda_k\mathfrak{g}_i)$ (resp. $\mathfrak{g}_i=\oplus_{\kappa\in\mathbf{Q}}({}^\mu_\kappa\mathfrak{g}_i)$) for any $i\in\mathbf{Z}/m$ (we now use that $\lambda\in Y_{G_{\underline{0}}}$ (resp. $\mu\in Y_{G_{\underline{0}},\mathbf{Q}}$)).

Let $s\in\mathbf{Z}-\{0\}$. We show:

(a) *The subspace* $\mathfrak{g}^{(1)}:=\oplus_{k\in s\mathbf{Z}}({}^\lambda_k\mathfrak{g}_{\underline{k/s}})$ *of* $\mathfrak{g}$ *is the Lie algebra of a well-defined connected reductive subgroup* $G^{(1)}$ *of* G.

We apply 1.10(a) to $H=G,\mathfrak{h}=\mathfrak{g}$ with the $\mathbf{Z}$-grading $\mathfrak{g}=\oplus_k({}^\lambda_k\mathfrak{g})$. We see that there is a well-defined reductive connected subgroup $H^{(1)}$ of G whose Lie algebra is $\mathfrak{h}^{(1)}=\oplus_{k\in s\mathbf{Z}}({}^\lambda_k\mathfrak{g})$. Note that $H^{(1)}$ contains $\lambda(\mathbf{k}^*)$ and is ϑ-stable. We choose $\zeta'\in\mathbf{k}^*$ such that $\zeta'^s=\zeta$. We define $\omega:H^{(1)}\to H^{(1)}$ by $\omega(h)=\mathrm{Ad}(\lambda(\zeta'))^{-1}\vartheta(h)$; this is an automorphism of $H^{(1)}$. The automorphism $\omega':\mathfrak{h}^{(1)}\to\mathfrak{h}^{(1)}$ induced by ω sends $x\in{}^\lambda_k\mathfrak{g}_i$ (where $k\in s\mathbf{Z}$, $i\in\mathbf{Z}/m$) to $\zeta'^{-k}\zeta^i x=\zeta^{i-\underline{k/s}}x$. Hence $\omega'^m=1$ and $\mathfrak{g}^{(1)}$ is equal to $\{x\in\mathfrak{h}^{(1)};\omega'(x)=x\}$. Let $G^{(1)}$ be the identity component of $\{h\in H^{(1)};\omega(h)=h\}$. Then $G^{(1)}$ is a connected reductive subgroup of $H^{(1)}$ with Lie algebra $\mathfrak{g}^{(1)}$. This proves (a).

Now ${}^\lambda_0\mathfrak{g}_{\underline{0}}$ is a Levi subalgebra of a parabolic subalgebra of $\mathfrak{g}_{\underline{0}}$. Hence $e^{{}^\lambda_0\mathfrak{g}_{\underline{0}}}$ is a well-defined subgroup of $G_{\underline{0}}$ (a Levi subgroup of a parabolic subgroup of $G_{\underline{0}}$). We have

(b) $e^{{}^\lambda_0\mathfrak{g}_{\underline{0}}}\subset G^{(1)}$.

2.5. **The definition of ϵ-spirals.** In the rest of this section we fix $\epsilon\in\{1,-1\}$. For any $\mu\in Y_{G_{\underline{0}},\mathbf{Q}}$ and any $N\in\mathbf{Z}$ we set

(a) $$ {}^\epsilon\mathfrak{p}^\mu_N=\oplus_{\kappa\in\mathbf{Q};\kappa\ge N\epsilon}({}^\mu_\kappa\mathfrak{g}_{\underline{N}}).$$

If $r\in\mathbf{Z}_{>0}$ is such that $\lambda:=r\mu\in Y_{G_{\underline{0}}}$ then we have

$$ {}^\epsilon\mathfrak{p}^\mu_N=\oplus_{k\in\mathbf{Z};k\ge rN\epsilon}({}^\lambda_k\mathfrak{g}_{\underline{N}}).$$

A collection $\{\mathfrak{p}_N;N\in\mathbf{Z}\}$ (or $\mathfrak{p}_*$) of subspaces of $\mathfrak{g}$ is said to be an ϵ-*spiral* if there exists $\mu\in Y_{G_{\underline{0}},\mathbf{Q}}$ such that $\mathfrak{p}_N={}^\epsilon\mathfrak{p}^\mu_N$ for any $N\in\mathbf{Z}$. We then set (for $N\in\mathbf{Z}$)

$$\mathfrak{u}_N=\{x\in\mathfrak{g}_{\underline{N}};\langle x,{}^\epsilon\mathfrak{p}^\mu_{-N}\rangle=0\}=\oplus_{\kappa\in\mathbf{Q};\kappa>N\epsilon}({}^\mu_\kappa\mathfrak{g}_{\underline{N}}).$$

We say that $\mathfrak{u}_*=\{\mathfrak{u}_N;N\in\mathbf{Z}\}$ is the nilradical of $\mathfrak{p}_*$.

The following properties of $\mathfrak{p}_*, \mathfrak{u}_*$ are immediate:

$\ldots \subset \mathfrak{p}_N \subset \mathfrak{p}_{N-\epsilon m} \subset \mathfrak{p}_{N-2\epsilon m} \subset \ldots$ for any N;
$\mathfrak{p}_N \subset \mathfrak{g}_{\underline{N}}$ for any N; $\mathfrak{p}_N = 0$ if $N\epsilon \gg 0$; $\mathfrak{p}_N = \mathfrak{g}_{\underline{N}}$ if $N\epsilon \ll 0$;
$[\mathfrak{p}_N, \mathfrak{p}_{N'}] \subset \mathfrak{p}_{N+N'}$ for any N, N' in $\mathbf{Z}$;
$\ldots \subset \mathfrak{u}_N \subset \mathfrak{u}_{N-\epsilon m} \subset \mathfrak{u}_{N-2\epsilon m} \subset \ldots$ for any N;
$\mathfrak{u}_N \subset \mathfrak{p}_N$ for any N; $\mathfrak{u}_N = \mathfrak{g}_{\underline{N}}$ if $N\epsilon \ll 0$;
$[\mathfrak{u}_N, \mathfrak{p}_{N'}] \subset \mathfrak{u}_{N+N'}$ for any N, N' in $\mathbf{Z}$.

For $N \in \mathbf{Z}$ we set $\mathfrak{l}_N = \mathfrak{p}_N/\mathfrak{u}_N$ and $\mathfrak{l} = \oplus_{N\in\mathbf{Z}}\mathfrak{l}_N$. We have $\mathfrak{l}_N = 0$ if $N \gg 0$ or if $N \ll 0$ hence $\dim \mathfrak{l} < \infty$; moreover, $[,] : \mathfrak{p}_N \times \mathfrak{p}_{N'} \to \mathfrak{p}_{N+N'}$ induces an operation $\mathfrak{l}_N \times \mathfrak{l}_{N'} \to \mathfrak{l}_{N+N'}$ which defines a Lie algebra structure on $\mathfrak{l}$.

Note that $\mathfrak{p}_0$ is a parabolic subagebra of the reductive Lie algebra $\mathfrak{g}_{\underline{0}}$ and $\mathfrak{u}_0 = \{x \in \mathfrak{g}_{\underline{0}}; \langle x, \mathfrak{p}_0\rangle = 0\}$ is the nilradical of $\mathfrak{p}_0$. We set $P_0 = e^{\mathfrak{p}_0} \subset G_{\underline{0}}$, $U_0 = e^{\mathfrak{u}_0} \subset G_{\underline{0}}$. Then P_0 is a parabolic subgroup of $G_{\underline{0}}$ and $U_0 = U_{P_0}$, so that $L_0 := P_0/U_0$ is a connected reductive group. We note that:

(b) *The* Ad-*action of* P_0 *on* $\mathfrak{g}$ *leaves stable* $\mathfrak{p}_N$ *and* $\mathfrak{u}_N$ *for any* N.

From (b) we see that for any N there is an induced action of P_0 on $\mathfrak{l}_N = \mathfrak{p}_N/\mathfrak{u}_N$. We show:

(c) *The restriction of this action to* U_0 *is trivial.*

It is enough to show that the ad-action of $\mathfrak{u}_0$ on $\mathfrak{p}_N/\mathfrak{u}_N$ is zero. This follows from the inclusion $[\mathfrak{u}_0, \mathfrak{p}_N] \subset \mathfrak{u}_N$ which has been noted earlier.

From (b),(c) we see that for any N there is an induced action of $L_0 = P_0/U_0$ on $\mathfrak{l}_N = \mathfrak{p}_N/\mathfrak{u}_N$. We show:

(d) *if* $x \in \mathfrak{p}_N$, $N\epsilon > 0$, *then* $x \in \mathfrak{g}^{nil}_{\underline{N}}$.

It is enough to show that for any $x' \in \mathfrak{g}$ we have $\mathrm{ad}(x)^n(x') = 0$ for $n \gg 0$. We can assume that $x' \in \mathfrak{g}_i$ for some $i \in \mathbf{Z}/m$. If $N' \in \mathbf{Z}$ satisfies $\underline{N'} = i$ and $N'\epsilon \ll 0$, then $\mathfrak{p}_{N'} = \mathfrak{g}_i$; thus we have $x' \in \mathfrak{p}_{N'}$ for some N'. We have $\mathrm{ad}(x)x' = [x, x'] \in \mathfrak{p}_{N+N'}$, $\mathrm{ad}(x)^2(x') \in \mathfrak{p}_{2n+N'}$ and, more generally, $\mathrm{ad}(x)^n(x') \in \mathfrak{p}_{nN+N'}$ for $n \geq 1$. If $n \gg 0$ we have $nN\epsilon + N'\epsilon \gg 0$ hence $\mathfrak{p}_{nN+N'} = 0$; thus, $\mathrm{ad}(x)^n(x') = 0$. This proves (d).

An element $\mu \in Y_{G_{\underline{0}},\mathbf{Q}}$ is said to be *p-regular* if $r\mu \in Y_{G_{\underline{0}}}$ for some $r \in \mathbf{Z}_{>0}$ such that $r \notin p\mathbf{Z}$. (This condition holds automatically if $p = 0$.) An ϵ-spiral $\mathfrak{p}_*$ is said to be p-reqular if $\mathfrak{p}_* = {}^{\epsilon}\mathfrak{p}^{\mu}_*$ for some p-regular $\mu \in Y_{G_{\underline{0}},\mathbf{Q}}$.

2.6. **Splittings of ϵ-spirals.** For $\mu \in Y_{G_{\underline{0}},\mathbf{Q}}$ and $N \in \mathbf{Z}$ we set

$$ {}^{\epsilon}\tilde{\mathfrak{l}}^{\mu}_N = \oplus_{\kappa\in\mathbf{Q};\kappa=N\epsilon}({}^{\mu}_{\kappa}\mathfrak{g}_{\underline{N}}) = {}^{\mu}_{N\epsilon}\mathfrak{g}_{\underline{N}}. $$

If $r \in \mathbf{Z}_{>0}$ is such that $\lambda := r\mu \in Y_{G_{\underline{0}}}$, then we have

$$ {}^{\epsilon}\tilde{\mathfrak{l}}^{\mu}_N = {}^{\lambda}_{rN\epsilon}\mathfrak{g}_{\underline{N}}. $$

A *splitting* of an ϵ-spiral $\mathfrak{p}_*$ is a collection $\{\tilde{\mathfrak{l}}_N; N \in \mathbf{Z}\}$ (or $\tilde{\mathfrak{l}}_*$) of subspaces of $\mathfrak{g}$ such that for some $\mu \in Y_{G_{\underline{0}},\mathbf{Q}}$ we have $\mathfrak{p}_* = {}^{\epsilon}\mathfrak{p}^{\mu}_*$ and $\tilde{\mathfrak{l}}_N = {}^{\epsilon}\tilde{\mathfrak{l}}^{\mu}_N$ for any $N \in \mathbf{Z}$. Let $\mathfrak{u}_*$ be the nilradical of $\mathfrak{p}_*$. From the definitions we see that $\mathfrak{p}_N = \mathfrak{u}_N \oplus \tilde{\mathfrak{l}}_N$ for any N, $[\tilde{\mathfrak{l}}_N, \tilde{\mathfrak{l}}_{N'}] \subset \tilde{\mathfrak{l}}_{N+N'}$ for any N, N' and the sum $\tilde{\mathfrak{l}} := \sum_{N\in\mathbf{Z}} \tilde{\mathfrak{l}}_N$ (in $\mathfrak{g}$) is direct. Now $\tilde{\mathfrak{l}}$ is a Lie subalgebra of $\mathfrak{g}$ which is $\mathbf{Z}$-graded by the subspaces $\tilde{\mathfrak{l}}_N$. Note that the isomorphisms $\tilde{\mathfrak{l}}_N \xrightarrow{\sim} \mathfrak{l}_N$ (restrictions of the obvious maps $\mathfrak{p}_N \to \mathfrak{l}_N$) give rise after taking $\oplus_N$ to an isomorphism $\tilde{\mathfrak{l}} \xrightarrow{\sim} \mathfrak{l}$ which is compatible with the Lie algebra structures and the $\mathbf{Z}$-gradings.

For μ as above we can find $\lambda \in Y_{G_{\underline{0}}}$ and $r \in \mathbf{Z}_{>0}$ such that $r\mu = \lambda$. Applying 2.4(a) with $s = r\epsilon$ we see that:

(a) *There is a well-defined connected reductive subgroup $\tilde{L}$ of G whose Lie algebra is $\tilde{\mathfrak{l}}$. In particular, $\tilde{\mathfrak{l}}$ and $\mathfrak{l}$ are reductive Lie algebras.*

Let $\tilde{L}_0 = e^{\tilde{\mathfrak{l}}_0}$. From 2.4(b) we have:

(b) $\tilde{L}_0 \subset \tilde{L}$.

We show:

(c) *Assume that we have $\tilde{\mathfrak{l}}_* = {}^\epsilon\tilde{\mathfrak{l}}^\mu_*$, $\mathfrak{p}_* = {}^\epsilon\mathfrak{p}^\mu_*$ where μ is p-regular, that is, $\mu = r\lambda$ with $\lambda \in Y_{G_{\underline{0}}}$ and $r \in \mathbf{Z}_{>0}$ such that $r \notin p\mathbf{Z}$. Then there exists ζ', a root of 1 in $\mathbf{k}^*$ such that $\tilde{\mathfrak{l}} = \{x \in \mathfrak{g}; \mathrm{Ad}(\lambda(\zeta')^{-1})\theta(x)) = x\}$, $\tilde{L} = G^{\mathrm{Ad}(\lambda(\zeta')^{-1})\vartheta} = e^{\tilde{\mathfrak{l}}} \subset G$; note that $\tilde{L} \in \mathfrak{G}$.*

Let ζ' be a primitive root of 1 of order rm in $\mathbf{k}^*$ such that $\zeta'^{r\epsilon} = \zeta$. We have $\mathfrak{g} = \oplus_{k\in\mathbf{Z},i\in\mathbf{Z}/m}({}^\lambda_k\mathfrak{g}_i)$, $\tilde{\mathfrak{l}}_N = {}^\lambda_{Nr\epsilon}\mathfrak{g}_{\underline{N}}$ for all $N \in \mathbf{Z}$. For $k, N' \in \mathbf{Z}$ and $x \in {}^\lambda_k\mathfrak{g}_{\underline{N'}}$ we have

$$\mathrm{Ad}(\lambda(\zeta')^{-1})(\theta(x)) = \zeta'^{-k}\zeta^{N'}x = \zeta'^{rN'\epsilon-k}x.$$

The condition that $\zeta'^{rN'\epsilon-k} = 1$ is that $rN'\epsilon - k \in rm\mathbf{Z}$ or that $k \in r\mathbf{Z}$ and $\underline{N'} = \underline{k/(r\epsilon)}$. We see that

$$\{x \in \mathfrak{g}; \mathrm{Ad}(\lambda(\zeta')^{-1})(\theta(x)) = x\} = \oplus_{k\in r\mathbf{Z},i\in\mathbf{Z}/m;\underline{k/(r\epsilon)}=i}({}^\lambda_k\mathfrak{g}_i) = \oplus_{N\in\mathbf{Z}}({}^\lambda_{rN\epsilon}\mathfrak{g}_{\underline{N}}) = \tilde{\mathfrak{l}},$$

and (c) follows.

We return to the general case.

We have $\lambda(\mathbf{k}^*) \subset \tilde{L}_0$; moreover, $\mathrm{Ad}(\lambda(t))$ acts as identity on $\tilde{\mathfrak{l}}_0 = {}^\lambda_0\mathfrak{g}_0 = \mathfrak{L}\tilde{L}_0$; thus, $\lambda(\mathbf{k}^*) \subset \mathcal{Z}_{\tilde{L}_0}$. Since $\mathbf{k}^*$ is connected, we deduce:

(d) $\lambda(\mathbf{k}^*) \subset \mathcal{Z}^0_{\tilde{L}_0}$.

Note that:

(e) *For $t \in \mathbf{k}^*, N \in \mathbf{Z}$, $\mathrm{Ad}(\lambda(t))$ acts on $\mathfrak{l}_N$ as $t^{rN\epsilon}$ times identity.*

We show:

(f) *If $\tilde{\mathfrak{l}}_*$ is a splitting of an ϵ-spiral $\mathfrak{p}_*$, then $\tilde{\mathfrak{l}}_*$ is a splitting of an $(-\epsilon)$-spiral.*

Let $\mu \in Y_{G_{\underline{0}},\mathbf{Q}}$ be such that $\tilde{\mathfrak{l}}_* = {}^\epsilon\tilde{\mathfrak{l}}^\mu_*$, $\mathfrak{p}_* = {}^\epsilon\mathfrak{p}^\mu_*$. Let $\mu' = (-1)\mu \in Y_{G_{\underline{0}},\mathbf{Q}}$. Then $\tilde{\mathfrak{l}}_* = {}^{-\epsilon}\tilde{\mathfrak{l}}^{\mu'}_*$ is a splitting of the $(-\epsilon)$-spiral ${}^{-\epsilon}\mathfrak{p}^{\mu'}_*$.

2.7. Let $\mathfrak{S}$ be the set of splittings of an ϵ-spiral $\mathfrak{p}_*$. Clearly, $\mathfrak{S} \neq \emptyset$. Let U_0 be as in 2.5. Now U_0 acts on $\mathfrak{S}$ by $u : \tilde{\mathfrak{l}}_* \mapsto \{\mathrm{Ad}(u)\tilde{\mathfrak{l}}_N; N \in \mathbf{Z}\}$. (We use that $\mathrm{Ad}(u)\mathfrak{p}_N = \mathfrak{p}_N$ for any N.) We show:

(a) *This U_0-action on $\mathfrak{S}$ is simply transitive.*

Let $\mathfrak{u}_*$ be the nilradical of $\mathfrak{p}_*$. Let $\tilde{\mathfrak{l}}_* \in \mathfrak{S}$, $\tilde{\mathfrak{l}}'_* \in \mathfrak{S}$. Since $\tilde{\mathfrak{l}}_0, \tilde{\mathfrak{l}}'_0$ are Levi subalgebras of $\mathfrak{p}_0$, there is a unique $u \in U_0$ such that $\mathrm{Ad}(u)\tilde{\mathfrak{l}}_0 = \tilde{\mathfrak{l}}'_0$. It remains to show that this u satisfies $\mathrm{Ad}(u)\tilde{\mathfrak{l}}_N = \tilde{\mathfrak{l}}'_N$ for any N. Let $\tilde{\mathfrak{l}} = \oplus_N\tilde{\mathfrak{l}}_N$, $\tilde{\mathfrak{l}}' = \oplus_N\tilde{\mathfrak{l}}'_N$ (these are Lie subalgebras of $\mathfrak{g}$) and let $\tilde{L} = e^{\tilde{\mathfrak{l}}} \subset G$, $\tilde{L}' = e^{\tilde{\mathfrak{l}}'} \subset G$. Let μ, μ' in $Y_{G_{\underline{0}},\mathbf{Q}}$ be such that $\mathfrak{p}_* = {}^\epsilon\mathfrak{p}^\mu_* = {}^\epsilon\mathfrak{p}^{\mu'}_*$, $\tilde{\mathfrak{l}}_* = {}^\epsilon\tilde{\mathfrak{l}}^\mu_*$, $\tilde{\mathfrak{l}}'_* = {}^\epsilon\tilde{\mathfrak{l}}^{\mu'}_*$. We can find $r \in \mathbf{Z}_{>0}$ such that $\lambda := r\mu \in Y_{G_{\underline{0}}}$, $\lambda' := r\mu' \in Y_{G_{\underline{0}}}$. Let $\tilde{L}_0$ be as in 2.6 and let $\tilde{L}'_0$ be the analogous subgroup of $\tilde{L}'$. We now fix $N \in \mathbf{Z}$. The Ad-action of $\tilde{L}_0$ (resp. $\tilde{L}'_0$) on $\mathfrak{g}$ leaves stable $\tilde{\mathfrak{l}}_N, \mathfrak{u}_N$ (resp. $\tilde{\mathfrak{l}}'_N, \mathfrak{u}_N$). Let $\tilde{L}''_0 = u\tilde{L}_0u^{-1}$, $\tilde{\mathfrak{l}}''_N = \mathrm{Ad}(u)\tilde{\mathfrak{l}}_N$; then the Ad-action of $\tilde{L}''_0$ on $\mathfrak{g}$ leaves stable $\tilde{\mathfrak{l}}''_N, \mathfrak{u}_N$. Since $\mathrm{Ad}(u)\tilde{\mathfrak{l}}_0 = \tilde{\mathfrak{l}}'_0$, we have $u\tilde{L}_0u^{-1} = \tilde{L}'_0$ hence $\tilde{L}'_0 = \tilde{L}''_0$. Let T be a maximal torus of $\tilde{L}'_0 = \tilde{L}''_0$. Now the Ad-action of T on $\mathfrak{g}$ leaves stable

$\tilde{\mathfrak{l}}'_N, \tilde{\mathfrak{l}}''_N, \mathfrak{u}_N, \mathfrak{p}_N$. Let $\mathcal{X} = \mathrm{Hom}(T, \mathbf{k}^*)$. For any $\alpha \in \mathcal{X}$ let

$$\mathfrak{p}_{N,\alpha} = \{x \in \mathfrak{p}_N; \mathrm{Ad}(\tau)x = \alpha(\tau)x \quad \forall \tau \in T\}, \quad \mathfrak{u}_{N,\alpha} = \mathfrak{u}_N \cap \mathfrak{p}_{N,\alpha},$$

$$\tilde{\mathfrak{l}}'_{N,\alpha} = \tilde{\mathfrak{l}}'_N \cap \mathfrak{p}_{N,\alpha}, \quad \tilde{\mathfrak{l}}''_{N,\alpha} = \tilde{\mathfrak{l}}''_N \cap \mathfrak{p}_{N,\alpha}.$$

We have $\tilde{\mathfrak{l}}'_N = \oplus_{\alpha\in\mathcal{X}}\tilde{\mathfrak{l}}'_{N,\alpha}$, $\tilde{\mathfrak{l}}''_N = \oplus_{\alpha\in\mathcal{X}}\tilde{\mathfrak{l}}''_{N,\alpha}$, $\mathfrak{u}_N = \oplus_{\alpha\in\mathcal{X}}\mathfrak{u}_{N,\alpha}$. Let $\mathcal{R}' = \{\alpha \in \mathcal{X}; \tilde{\mathfrak{l}}'_{N,\alpha} \neq 0\}$, $\mathcal{R}'' = \{\alpha \in \mathcal{X}; \tilde{\mathfrak{l}}''_{N,\alpha} \neq 0\}$, $\tilde{\mathcal{R}} = \{\alpha \in \mathcal{X}; \mathfrak{u}_{N,\alpha} \neq 0\}$. Since $\tilde{\mathfrak{l}}'_N, \tilde{\mathfrak{l}}''_N$ are T-stable complements of $\mathfrak{u}_N$ in $\mathfrak{p}_N$, the T-modules $\tilde{\mathfrak{l}}'_N, \tilde{\mathfrak{l}}''_N$ are isomorphic, hence $\mathcal{R}' = \mathcal{R}''$. Since $\lambda'(\mathbf{k}^*) \subset \mathcal{Z}^0_{\tilde{L}'_0}$ (see 2.6(d)), we have $\lambda'(\mathbf{k}^*) \subset T$; hence for any $\alpha \in \mathcal{X}$ we can define $\alpha \bullet \lambda' \in \mathbf{Z}$ by $\alpha(\lambda'(t)) = t^{\alpha\bullet\lambda'}$ for all $t \in \mathbf{k}^*$.

Assume that $\alpha \in \tilde{\mathcal{R}}$. Then for any $t \in \mathbf{k}^*$, $\mathrm{Ad}(\lambda'(t))$ acts on $\mathfrak{u}_{N,\alpha}$ as multiplication by $t^{\alpha\bullet\lambda'}$ hence $\mathfrak{u}_{N,\alpha} \subset {}^{\lambda'}_{\alpha\bullet\lambda'}\mathfrak{g}_{\underline{N}}$; thus ${}^{\lambda'}_{\alpha\bullet\lambda'}\mathfrak{g}_{\underline{N}}$ has a nonzero intersection with $\mathfrak{u}_N$, so that $\alpha \bullet \lambda' > rN\epsilon$. We see that $\tilde{\mathcal{R}} \subset \{\alpha \in \mathcal{X}; \alpha \bullet \lambda' > rN\epsilon\}$. Assume now that $\alpha \in \mathcal{R}'$. Then for any $t \in \mathbf{k}^*$, $\mathrm{Ad}(\lambda'(t))$ acts on $\mathfrak{l}'_{N,\alpha}$ as multiplication by $t^{\alpha\bullet\lambda'}$ hence $\mathfrak{l}'_{N,\alpha} \subset {}^{\lambda'}_{\alpha\bullet\lambda'}\mathfrak{g}_{\underline{N}}$; thus, ${}^{\lambda'}_{\alpha\bullet\lambda'}\mathfrak{g}_{\underline{N}}$ has a nonzero intersection with $\tilde{\mathfrak{l}}'_N$, so that $\alpha \bullet \lambda' = rN\epsilon$. We see that $\mathcal{R}' \subset \{\alpha \in \mathcal{X}; \alpha \bullet \lambda' = rN\epsilon\}$. It follows that $\mathcal{R}' \cap \tilde{\mathcal{R}} = \emptyset$ so that $\mathfrak{p}_{N,\alpha} = \tilde{\mathfrak{l}}'_{N,\alpha}$ for $\alpha \in \mathcal{R}'$. Since $\mathcal{R}' = \mathcal{R}''$, we have also $\mathcal{R}'' \cap \tilde{\mathcal{R}} = \emptyset$, so that $\mathfrak{p}_{N,\alpha} = \tilde{\mathfrak{l}}''_{N,\alpha}$ for $\alpha \in \mathcal{R}'' = \mathcal{R}'$. Thus, for $\alpha \in \mathcal{R}' = \mathcal{R}''$ we have $\tilde{\mathfrak{l}}'_{N,\alpha} = \tilde{\mathfrak{l}}''_{N,\alpha}$ hence $\tilde{\mathfrak{l}}'_N = \tilde{\mathfrak{l}}''_N$ and $\tilde{\mathfrak{l}}'_N = \mathrm{Ad}(u)\tilde{\mathfrak{l}}_N$. This proves (a).

For any splitting $\tilde{\mathfrak{l}}_*$ of $\mathfrak{p}_*$ we denote by $\tilde{L}(\tilde{\mathfrak{l}}_*)$ the connected reductive subgroup $\tilde{L}$ of G associated to $\tilde{\mathfrak{l}}_*$ in 2.6. The family of groups $(\tilde{L}(\tilde{\mathfrak{l}}_*))$ indexed by the various splittings $\tilde{\mathfrak{l}}_*$ of $\mathfrak{p}_*$ has the property that any two groups in the family are canonically isomorphic to each other; the isomorphism is provided by conjugation by a well-defined $u \in U_0$ (this follows from (a)). It follows that the groups in the family can be identified with a single connected reductive group L which is canonically isomorphic to each group in the family. Note that L is canonically attached to the ϵ-spiral $\mathfrak{p}_*$ and that $\mathfrak{L}L = \mathfrak{l}$ canonically. Note also that L_0 in 2.5 is naturally a closed subgroup of L.

2.8. **Subspirals coming from parabolics of $\mathfrak{l}_*$.** Let $\mathfrak{p}_*$ be an ϵ-spiral. We define $\mathfrak{u}_*$, $\mathfrak{l}_*$, $\mathfrak{l}$ in terms of $\mathfrak{p}_*$ as in 2.5. Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{l}$ compatible with the $\mathbf{Z}$-grading of $\mathfrak{l}$ that is, such that $\mathfrak{q} = \oplus_{N\in\mathbf{Z}}\mathfrak{q}_N$ where $\mathfrak{q}_N = \mathfrak{q} \cap \mathfrak{l}_N$. For any $N \in \mathbf{Z}$ let $\hat{\mathfrak{p}}_N$ be the inverse image of $\mathfrak{q}_N$ under the obvious map $\mathfrak{p}_N \to \mathfrak{l}_N$. We show:

(a) *$\hat{\mathfrak{p}}_*$ is an ϵ-spiral. Moreover, if $\mathfrak{p}_*$ is p-regular then $\hat{\mathfrak{p}}_*$ is p-regular.*

We can find $\mu \in Y_{G_{\underline{0}},\mathbf{Q}}$ such that $\mathfrak{p}_* = {}^{\epsilon}\mathfrak{p}^{\mu}_*$; let $\tilde{\mathfrak{l}}_* = {}^{\epsilon}\tilde{\mathfrak{l}}^{\mu}_*$. Let $\tilde{L}$ be as in 2.6. Let $\tilde{\mathfrak{q}}$ be the Lie subalgebra of $\tilde{\mathfrak{l}}$ corresponding to $\mathfrak{q}$ under the obvious isomorphism $\tilde{\mathfrak{l}} \xrightarrow{\sim} \mathfrak{l}$ and let $\tilde{\mathfrak{q}}_N = \tilde{\mathfrak{q}} \cap \tilde{\mathfrak{l}}_N$ so that $\tilde{\mathfrak{q}} = \oplus_{N\in\mathbf{Z}}\tilde{\mathfrak{q}}_N$. We then have $\hat{\mathfrak{p}}_N = \mathfrak{u}_N \oplus \tilde{\mathfrak{q}}_N$ for all N. Let $r \in \mathbf{Z}_{>0}$ be such that $\lambda := r\mu \in Y_{G_{\underline{0}}}$; if $\mathfrak{p}_*$ is p-regular we assume in addition that $r \notin p\mathbf{Z}$.

From 2.6(e) we see that for $t \in \mathbf{k}^*$, $\mathrm{Ad}(\lambda(t))$ leaves stable each $\tilde{\mathfrak{q}}_N$ hence it leaves stable $\tilde{\mathfrak{q}}$. It follows that $\mathbf{k}^*$ acts via $t \mapsto \mathrm{Ad}(\lambda(t))$ on the variety of Levi subalgebras of $\tilde{\mathfrak{q}}$; since this variety is isomorphic to an affine space, there exists a Levi subalgebra $\mathfrak{m}$ of $\tilde{\mathfrak{q}}$ such that $\mathrm{Ad}(\lambda(t))\mathfrak{m} = \mathfrak{m}$ for all $t \in \mathbf{k}^*$. Let R be the closed connected subgroup of $\tilde{L}$ (a torus) such that $\mathfrak{L}R$ is the center of $\mathfrak{m}$. Since $\tilde{\mathfrak{q}}$ is a parabolic subalgebra of $\tilde{\mathfrak{l}}$ with Levi subalgebra $\mathfrak{m}$, we can find $\lambda' \in Y_R$ such that, setting for any $N' \in \mathbf{Z}$:

$$ {}^{\lambda'}_{N'}\tilde{\mathfrak{l}} = \{x \in \tilde{\mathfrak{l}}; \mathrm{Ad}(\lambda'(t))x = t^{N'}x \quad \forall t \in \mathbf{k}^*\}, $$

we have $\tilde{\mathfrak{q}} = \oplus_{N' \in \mathbf{Z}_{\geq 0}}({}^{\lambda'}_{N'}\tilde{\mathfrak{l}})$, $\mathfrak{m} = {}^{\lambda'}_{0}\tilde{\mathfrak{l}}$. We have $\mathfrak{m} = \oplus_N \mathfrak{m}_N$ where $\mathfrak{m}_N = \mathfrak{m} \cap \tilde{\mathfrak{l}}_N$ and $\mathfrak{m}_0$ is a Levi subalgebra of a parabolic subalgebra of $\mathfrak{m}$. Hence a Cartan subalgebra of $\mathfrak{m} \cap \tilde{\mathfrak{l}}_0$ is also a Cartan subalgebra of $\mathfrak{m}$, so that it contains the center of $\mathfrak{m}$. Thus the center of $\mathfrak{m}$ is contained in $\tilde{\mathfrak{l}}_0$, so that $R \subset \tilde{L}_0$. Since for any $t, t' \in \mathbf{k}^*$, $\lambda(t)$ is contained in $\mathcal{Z}_{\tilde{L}_0}$ and $\lambda'(t') \in \tilde{L}_0$, we have $\lambda(t)\lambda'(t') = \lambda'(t')\lambda(t)$. We can view λ' as an element of $Y_{G_{\underline{0}}}$ hence ${}^{\lambda'}_{k}\mathfrak{g}_i$ is defined for $k \in \mathbf{Z}, i \in \mathbf{Z}/m$ and we have $\mathfrak{g}_i = \oplus_{k \in \mathbf{Z}}({}^{\lambda'}_{k}\mathfrak{g}_i)$ for any $i \in \mathbf{Z}/m$. We can find $a \in \mathbf{Z}_{>0}$ such that ${}^{\lambda'}_{k}\mathfrak{g}_i = 0$ for any $i \in \mathbf{Z}/m$ and any $k \in \mathbf{Z} - [-a, a]$. Let b be an integer such that $b > 2a$, $b \notin p\mathbf{Z}$. We define $\lambda'' \in Y_{G_{\underline{0}}}$ by $\lambda''(t) = \lambda(t^b)\lambda'(t) = \lambda'(t)\lambda(t^b)$ for all $t \in \mathbf{k}^*$. By definition, for $k \in \mathbf{Z}, i \in \mathbf{Z}/m$ we have:

$$ \begin{aligned} {}^{\lambda''}_{k}\mathfrak{g}_i &= \{x \in \mathfrak{g}_i; \mathrm{Ad}(\lambda(t^b)\lambda'(t))x = t^k x \quad \forall t \in \mathbf{k}^*\} \\ &= \oplus_{k', k_2; k' \in b\mathbf{Z}, k_2 \in \mathbf{Z}, k' + k_2 = k}({}^{\lambda}_{k'/b}\mathfrak{g}_i \cap {}^{\lambda'}_{k_2}\mathfrak{g}_i). \end{aligned} $$

When ${}^{\lambda''}_{k}\mathfrak{g}_i \neq 0$ then $k = bk_1 + k_2$ for some $k_1 \in \mathbf{Z} \cap [-a, a]$, $k_2 \in \mathbf{Z}$; in this case, k_1, k_2 are uniquely determined by k since $b > 2a$. Thus, we have

$$ {}^{\lambda''}_{k}\mathfrak{g}_i = {}^{\lambda}_{k_1}\mathfrak{g}_i \cap {}^{\lambda'}_{k_2}\mathfrak{g}_i \text{ if } k = bk_1 + k_2 \text{ with } k_1, k_2 \text{ in } \mathbf{Z}, $$

$$ {}^{\lambda''}_{k}\mathfrak{g}_i = 0, \text{ otherwise.} $$

Let $\mu' = \frac{1}{br}\lambda'' \in Y_{G_{\underline{0}}, \mathbf{Q}}$ and let $\mathfrak{p}'_* = {}^{\epsilon}\mathfrak{p}^{\mu'}_*$. For $N \in \mathbf{Z}$ we have

$$ \mathfrak{p}'_N = \oplus_{k_1, k_2 \in \mathbf{Z}; bk_1 + k_2 \geq Nbr\epsilon, |k_2| \leq a}({}^{\lambda}_{k_1}\mathfrak{g}_{\underline{N}} \cap {}^{\lambda'}_{k_2}\mathfrak{g}_{\underline{N}}). $$

The only integer multiple of b in $[-a, a]$ is 0; hence the condition that $k_2 \geq b(rN\epsilon - k_1)$ (with $k_2 \in [-a, a]$) is equivalent to the condition that either $0 > b(rN\epsilon - k_1)$, $k_2 \in [-a, a]$ or that $0 = b(rN\epsilon - k_1)$, $k_2 \in [0, a]$. Thus, $\mathfrak{p}'_N = X \oplus X'$, where

$$ X = \oplus_{k_1, k_2 \in \mathbf{Z}; k_1 > rN\epsilon}({}^{\lambda}_{k_1}\mathfrak{g}_{\underline{N}} \cap {}^{\lambda'}_{k_2}\mathfrak{g}_{\underline{N}}) = \oplus_{k_1 \in \mathbf{Z}; k_1 > rN\epsilon}({}^{\lambda}_{k_1}\mathfrak{g}_{\underline{N}}) = \mathfrak{u}_N, $$

$$ X' = \oplus_{k_1, k_2 \in \mathbf{Z}; k_1 = rN\epsilon, k_2 \geq 0}({}^{\lambda}_{k_1}\mathfrak{g}_{\underline{N}} \cap {}^{\lambda'}_{k_2}\mathfrak{g}_{\underline{N}}) = \tilde{\mathfrak{l}}_N \cap (\oplus_{k_2 \in \mathbf{Z}_{\geq 0}}({}^{\lambda'}_{k_2}\mathfrak{g}_{\underline{N}})) = \tilde{\mathfrak{l}}_N \cap \tilde{\mathfrak{q}} = \tilde{\mathfrak{q}}_N. $$

Thus, we have $\mathfrak{p}'_N = \mathfrak{u}_N \oplus \tilde{\mathfrak{q}}_N = \hat{\mathfrak{p}}_N$. This proves (a).

From the computation in the previous proof we can extract the following:

(b) *the splitting* ${}^{\epsilon}\tilde{\mathfrak{l}}^{\mu'}_*$ *of the ϵ-spiral* $\hat{\mathfrak{p}}_* = {}^{\epsilon}\mathfrak{p}^{\mu'}_*$ *is equal to* $\mathfrak{m}_*$.

2.9. **The spiral attached to an element** $x \in \mathfrak{g}^{nil}_{\delta}$**.** *In the remainder of this paper we fix $\eta \in \mathbf{Z} - \{0\}$ such that $\underline{\eta} = \delta$.*

In this subsection we assume that $\epsilon = \dot{\eta}$; see 0.12. Let $x \in \mathfrak{g}^{nil}_{\delta}$. We associate to x an ϵ-spiral as follows. By 2.3(b), we can find $\phi = (e, h, f) \in J_\delta(x)$ such that $e = x$. Let $\iota = \iota_\phi \in Y_G$ be as in 1.1. Since the differential of ι is the linear map $\mathbf{k} \to \mathfrak{g}$, $z \mapsto zh \in \mathfrak{g}_0$, we have $\iota(\mathbf{k}^*) \subset G_{\underline{0}}$ so that ι can be viewed as an element of $Y_{G_{\underline{0}}}$. Then $\mathfrak{p}^{\phi}_* := {}^{\epsilon}\mathfrak{p}^{(|\eta|/2)\iota}_*$ is an ϵ-spiral with splitting $\tilde{\mathfrak{l}}^{\phi}_* := {}^{\epsilon}\tilde{\mathfrak{l}}^{(|\eta|/2)\iota}_*$. Note that for $N \in \mathbf{Z}$ we have

$$ \mathfrak{p}^{\phi}_N = \oplus_{k \in \mathbf{Z}; k \geq 2N\epsilon}({}^{\iota}_{k/|\eta|}\mathfrak{g}_{\underline{N}}), \quad \tilde{\mathfrak{l}}^{\phi}_N = {}^{\iota}_{2N/\eta}\mathfrak{g}_{\underline{N}} \text{ if } 2N/\eta \in \mathbf{Z}, \quad \tilde{\mathfrak{l}}^{\phi}_N = 0 \text{ if } 2N/\eta \notin \mathbf{Z}. $$

We show that:

(a) *The ϵ-spiral $\mathfrak{p}^{\phi}_*$ is p-regular; it depends only on x, not on ϕ.*

The p-regularity follows from the fact that $2 \notin p\mathbf{Z}$. We now prove the second statement of (a). By 2.3(b), another choice for ϕ must be of the form $u\phi$ where $u \in U_{G(x)^0} \cap G_{\underline{0}}$. Let $\iota' = \iota_{u\phi}$. For $t \in \mathbf{k}^*$ we have $\iota'(t) = u\iota(t)u^{-1}$ hence ${}^{\iota'}_k\mathfrak{g}_i = \mathrm{Ad}(u)({}^{\iota}_k\mathfrak{g}_i)$ for any $k \in \mathbf{Z}, i \in \mathbf{Z}/m$. It follows that for $N \in \mathbf{Z}$ we have $\mathfrak{p}_N^{u\phi} = \mathrm{Ad}(u)\mathfrak{p}_N^{\phi}$. To show that $\mathfrak{p}_N^{u\phi} = \mathfrak{p}_N^{\phi}$, it is enough to show that $\mathrm{Ad}(u)\mathfrak{p}_N^{\phi} = \mathfrak{p}_N^{\phi}$. It is enough to show:

$\mathrm{Ad}(u)({}^{\iota}_k\mathfrak{g}) \subset \oplus_{k';k' \ge k}({}^{\iota}_{k'}\mathfrak{g})$ for any $u \in G(x), k \in \mathbf{Z}$.

Let P be the parabolic subgroup of G such that $\mathfrak{L}P = \oplus_{k \in \mathbf{Z}; k \ge 0}({}^{\iota}_k\mathfrak{g})$. Clearly, $\mathrm{Ad}(g)({}^{\iota}_k\mathfrak{g}) \subset \oplus_{k';k' \ge k}({}^{\iota}_{k'}\mathfrak{g})$ for any $g \in P, k \in \mathbf{Z}$. Hence it is enough to note the known inclusion $G(x) \subset P$. This proves (a).

In view of (a) we will write $\mathfrak{p}_*^x$ instead of $\mathfrak{p}_*^{\phi}$, where ϕ is any element in $J_\delta(x)$; let $\mathfrak{u}_*^x$ be the nilradical of $\mathfrak{p}_*^x$. Now the splitting $\tilde{\mathfrak{l}}_*^{\phi}$ depends in general on ϕ. We set $\tilde{\mathfrak{l}}^{\phi} = \oplus_{N \in \mathbf{Z}}\tilde{\mathfrak{l}}_N^{\phi}$; this is a $\mathbf{Z}$-graded Lie subalgebra of $\mathfrak{g}$. Let $\tilde{L}^{\phi} = e^{\tilde{\mathfrak{l}}^{\phi}} \subset G$; we have $\tilde{L}^{\phi} \in \mathfrak{G}$. Let $\tilde{L}_0^{\phi} = e^{\tilde{\mathfrak{l}}_0^{\phi}} \subset \tilde{L}^{\phi}$. We show:

(b) *We have $x \in \tilde{\mathfrak{l}}_\eta^{\phi}$; more precisely, x belongs to $\overset{\circ}{\tilde{\mathfrak{l}}}{}_\eta^{\phi}$ (the open $\tilde{L}_0^{\phi}$-orbit on $\tilde{\mathfrak{l}}_\eta^{\phi}$).*

The first statement is the same as $x \in {}^{\iota}_2\mathfrak{g}_\delta$; this follows from the equality $[h, x] = 2x$. The second statement can be deduced from [L4, 4.2(a)].

We set $\tilde{L}_0^{\phi}(x) = \tilde{L}_0^{\phi} \cap G(x)$, $G_{\underline{0}}(x) = G_{\underline{0}} \cap G(x)$. We show:

(c) *The inclusion $\tilde{L}_0^{\phi}(x) \to G_{\underline{0}}(x)$ induces an isomorphism on the groups of components.*

Let P_0 be the parabolic subgroup of $G_{\underline{0}}$ such that $\mathfrak{L}P_0 = \mathfrak{p}_0^x = \oplus_{k \in \mathbf{Z}; k \ge 0}({}^{\iota}_k\mathfrak{g}_{\underline{0}})$ and let $U_0 = U_{P_0}$. We set $P_0(x) = P_0 \cap G(x)$, $U_0(x) = U_0 \cap G(x)$. Then $\tilde{L}_0^{\phi}$ is a Levi subgroup of P_0 so that $P_0 = \tilde{L}_0^{\phi}U_0$ (semidirect product) and $P_0(x) = \tilde{L}_0^{\phi}(x)U_0(x)$ (semidirect product). Since $U_0(x)$ is a connected unipotent group we see that the inclusion $\tilde{L}_0^{\phi}(x) \to P_0(x)$ induces an isomorphism on the groups of components. It remains to show that $P_0(x) = G_{\underline{0}}(x)$. As we have noted in the proof of (a), we have $G(x) \subset P$ hence $G_{\underline{0}}(x) \subset P \cap G_{\underline{0}}$; since $P \cap G_{\underline{0}}$ and P_0 have the same Lie algebra, namely $\mathfrak{p}_0^x$, they must have the same identity component; since P_0 is parabolic in $G_{\underline{0}}$, we must have $P \cap G_{\underline{0}} = P_0$, so that $G_{\underline{0}}(x) \subset P_0$ and therefore $G_{\underline{0}}(x) \subset P_0(x)$. Since the reverse inclusion is obvious, we see that $P_0(x) = G_{\underline{0}}(x)$ and (c) is proved.

We show:

(d) *If $g \in G_{\underline{0}}$ is such that $\mathrm{Ad}(g^{-1})(x) \in \mathfrak{p}_\eta^x$, then $g \in P_0$.*

The assumption of (d) implies that $g \in P$. (We use [L4, 5.7] applied to the trivial $\mathbf{Z}$-grading of $\mathfrak{g}$ that is, the $\mathbf{Z}$-grading such that in [L4, 3.1] we have $\mathfrak{g}_N = 0$ for $N \ne 0$.) Thus, we have $g \in P \cap G_{\underline{0}}$. As in the proof of (c) we have $P \cap G_{\underline{0}} = P_0$ and (d) follows.

We show:

(e) *The P_0-orbit of x in $\mathfrak{p}_\eta^x$ is open dense in $\mathfrak{p}_\eta^x$.*

We argue as in [L4, 5.9]. It is enough to show that $\dim(P_0) - \dim(P_0 \cap G(x)) = \dim \mathfrak{p}_\eta^x$ or equivalently that

$$\dim \mathfrak{p}_0^x - \dim \ker(\mathrm{ad}(x) : \mathfrak{p}_0^x \to \mathfrak{g}_\delta) = \dim \mathfrak{p}_\eta^x.$$

Since $x \in \mathfrak{p}_\eta^x$ (see (b)) and $[\mathfrak{p}_0^x, \mathfrak{p}_\eta^x] \subset \mathfrak{p}_\eta^x$, we have $\mathrm{ad}(x)(\mathfrak{p}_0^x) \subset \mathfrak{p}_\eta^x$ so that it is enough to show that

$$\dim \ker(\mathrm{ad}(x) : \mathfrak{p}_0^x \to \mathfrak{p}_\eta^x) = \dim \mathfrak{p}_0^x - \dim \mathfrak{p}_\eta^x,$$

or equivalently, that $\mathrm{ad}(x) : \mathfrak{p}_0^x \to \mathfrak{p}_\eta^x$ is surjective. By the representation theory of $\mathfrak{sl}_2$, the linear map

$$\mathrm{ad}(x) : \oplus_{k\in\mathbf{Z};k\geq 0}({}^\iota_k\mathfrak{g}) \to \oplus_{k\in\mathbf{Z};k\geq 2}({}^\iota_k\mathfrak{g})$$

is surjective. This restricts for any $i \in \mathbf{Z}/m$ to a (necessarily surjective) map

$$\mathrm{ad}(x) : \oplus_{k\in\mathbf{Z};k\geq 0}({}^\iota_k\mathfrak{g}_i) \to \oplus_{k\in\mathbf{Z};k\geq 2}({}^\iota_k\mathfrak{g}_{i+\delta}).$$

Taking $i = 0$ we see that $\mathrm{ad}(x) : \mathfrak{p}_0^x \to \mathfrak{p}_\eta^x$ is surjective. This proves (e).

The assignment $x \mapsto \mathfrak{p}_*^x$ is a $\mathbf{Z}/m$-analogue of an assignment in the case of $\mathbf{Z}$-graded Lie algebras given in [L4, §5] which is in turn modelled on a construction in [KL, 7.1].

3. Admissible systems

In this section we introduce the set $\underline{\mathfrak{T}}_\eta$ of $G_{\underline{0}}$-conjugacy classes of admissible systems, which will be used to index the blocks in $\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_\delta^{nil})$. We also define a map that assigns a pair $(\mathcal{O}, \mathcal{L})$ (where $\mathcal{O}$ is a $G_{\underline{0}}$-orbit in $\mathfrak{g}_\delta^{nil}$ and $\mathcal{L}$ is an irreducible $G_{\underline{0}}$-equivariant local system on it) an element in $\underline{\mathfrak{T}}_\eta$.

3.1. **Definition of admissible systems.** We preserve the setup of 2.1.

Let $\mathfrak{T}'_\eta$ be the set consisting of all systems $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$, where $M \in \mathfrak{G}$, $\mathfrak{m} = \mathfrak{L}M$, $\mathfrak{m}_*$ is a $\mathbf{Z}$-grading of $\mathfrak{m}$, $M_0 = e^{\mathfrak{m}_0} \subset M$, $\tilde{C}$ is a simple cuspidal M_0-equivariant perverse sheaf on $\mathfrak{m}_\eta$ (up to isomorphism).

Until the end of 3.4 we fix $\dot{\xi} = (M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{T}'_\eta$. Let $\iota \in Y_M$ be associated to $\tilde{C}$ as in 1.2(c),(a) (with $M, \tilde{C}$ instead of H, A), so that ${}^\iota_k\mathfrak{m} = \mathfrak{m}_{\eta k/2}$ for any $k \in \mathbf{Z}$ such that $\eta k/2 \in \mathbf{Z}$ and ${}^\iota_k\mathfrak{m} = 0$ for any $k \in \mathbf{Z}$ such that $\eta k/2 \notin \mathbf{Z}$. Then we have $\mathfrak{m}_{k'} = {}^\iota_{2k'/\eta}\mathfrak{m}$ for $k' \in \mathbf{Z}$ such that $2k'/\eta \in \mathbf{Z}$ and $\mathfrak{m}_{k'} = 0$ for $k' \in \mathbf{Z}$ such that $2k'/\eta \notin \mathbf{Z}$. Note that $\iota(\mathbf{k}^*)$ is contained in $\mathcal{Z}^0_{M_0}$.

The system $\dot{\xi}$ is said to be *admissible* if conditions (a),(b) below are satisfied:

(a) we have $\mathfrak{m}_N \subset \mathfrak{g}_{\underline{N}}$ for any $N \in \mathbf{Z}$;

(b) there exists an element τ of finite order in the torus $\iota(\mathbf{k}^*)\mathcal{Z}^0_M$ of M_0 such that $M = G^{\mathrm{Ad}(\tau)\vartheta}$.

We now consider the following condition on $\dot{\xi}$ which may or may not hold.

(c) $\mathfrak{m}_*$ *is a splitting of some p-regular* 1*-spiral or, equivalently (see* 2.6(f)*), of some p-regular* (-1)*-spiral.*

The following result will be proved in 3.2–3.4.

(d) $\dot{\xi}$ *is admissible if and only if* $\dot{\xi}$ *satisfies* (c).

We now make some comments on the significance of condition (b). Assume that condition (a) is satisfied and that τ is any semisimple element of finite order of $G_{\underline{0}}$ such that $M = G^{\mathrm{Ad}(\tau)\vartheta}$. We show that we have automatically

(e) $$\tau \in \iota(\mathbf{k}^*)\mathcal{Z}_M.$$

Note that $\vartheta(\tau) = \tau$ since $\tau \in G_{\underline{0}}$ hence $\tau \in G^{\mathrm{Ad}(\tau)\vartheta} = M$.

Let $N \in \mathbf{Z}$ be such that $2N/\eta \in \mathbf{Z}$. Since $\mathfrak{m}_N \subset \mathfrak{g}_{\underline{N}}$, θ acts on $\mathfrak{m}_N$ as ζ^N; since $\mathrm{Ad}(\tau)\theta$ acts as 1 on $\mathfrak{m}$ we see that $\mathrm{Ad}(\tau)$ acts on $\mathfrak{m}_N$ as ζ^{-N}. On the other hand, for $t \in \mathbf{k}^*$, $\mathrm{Ad}(\iota(t))$ acts on $\mathfrak{m}_N$ as $t^{2N/\eta}$. Hence if $t_0 \in \mathbf{k}^*$ satisfies $t_0^{2/\eta} = \zeta^{-1}$, then we have $\mathrm{Ad}(\iota(t_0))\,\mathrm{Ad}(\tau^{-1}) = t_0^{2N/\eta}\zeta^N = \zeta^{-N}\zeta^N = 1$ on $\mathfrak{m}_N$. It follows that

$\mathrm{Ad}(\iota(t_0))\,\mathrm{Ad}(\tau^{-1}) = 1$ on $\mathfrak{m}$. Since $\iota(t_0)\tau^{-1} \in M$, we deduce that $\iota(t_0)\tau^{-1} \in \mathcal{Z}_M$ hence $\tau \in \iota(\mathbf{k}^*)\mathcal{Z}_M$, as asserted.

We see that condition (b) is a strengthening of (e) in which τ is required to lie not only in $\iota(\mathbf{k}^*)\mathcal{Z}_M$ but in its identity component.

3.2. We show:

(a) *For any element τ_0 of finite order in a torus T there exists $\lambda_0 \in Y_T$ such that $\tau_0 \in \lambda_0(\mathbf{k}^*)$.*

We can find $c \in \mathbf{Z}_{>0}$ such that $c \notin p\mathbf{Z}$ and $\tau_0^c = 1$. Let $\mu_c = \{z \in \mathbf{k}^*; z^c = 1\}$. For some $a \in \mathbf{N}$ we can identify $T = (\mathbf{k}^*)^a$ and τ_0 with $(z_1, \dots, z_a) \in (\mu_c)^a \subset T$. Now μ_c is cyclic with generator z_0. Thus we have $z_1 = z_0^{k_1}, \dots, z_a = z_0^{k_a}$, where $k_1, \dots, k_a$ are integers. We define $\lambda_0 \in Y_T$ by $t \mapsto (t^{k_1}, \dots, t^{k_a})$. Then $\tau_0 = \lambda_0(z_0)$, as desired.

We remark that in the proof of (a) we can assume that:

(b) $k_1 \in \mathbf{Z}_{>0}, k_1 \notin p\mathbf{Z}$.

Indeed, if $p = 0$, then $k_1 \notin p\mathbf{Z}$ is automatic. Assume now that $p > 0$. We write $k_1 = k_1' p^e$, where $k_1' \in \mathbf{Z} - p\mathbf{Z}$, $e \in \mathbf{Z}_{\geq 0}$. Define $z_0' \in \mu_c$ by $z_0' = z_0^{p^e}$. This is again a generator of μ_c. (Recall that $c \notin p\mathbf{Z}$.) We have $z_1 = (z_0')^{k_1'}$, $z_j = (z_0')^{k_j'}$, where $k_j' \in \mathbf{Z}_{>0}$ for $j = 2, 3, \dots, a$. Thus we can replace $z_0, k_1, \dots, k_s$ by $z_0', k_1', \dots, k_s'$, where $k_1' \in \mathbf{Z}_{>0}, k_1' \notin p\mathbf{Z}$. This proves (b).

We now assume that τ as in 3.1(b) is given. We show:

(c) *There exist $f \in \mathbf{Z}_{>0}$ and $\lambda' \in Y_{\mathcal{Z}_M^0}$ such that $f \notin p\mathbf{Z}$ and such that, if $\lambda \in Y_{\iota(\mathbf{k}^*)\mathcal{Z}_M^0}$ is defined by $\lambda(t) = \iota(t^f)\lambda'(t)$ for all t, then $\tau \in \lambda(\mathbf{k}^*)$.*

If ι is identically 1, then (c) follows from (a) applied to $T = \mathcal{Z}_M^0$ (we can take $f = 1$). Assume now that ι is not identically 1. Then $\iota : \mathbf{k}^* \to M$ has finite kernel. Let $T = \mathbf{k}^* \times \mathcal{Z}_M^0$; we define $d : T \to \iota(k^*)\mathcal{Z}_M^0$ by $d(t, g) = \iota(t)g$. By definition, $\iota(k^*)$ is contained in the derived subgroup of M hence it has finite intersection with $\mathcal{Z}_M^0$. It follows that d has finite kernel. It is also surjective, hence we can find $\tilde{\tau} \in T$ of finite order such that $d(\tilde{\tau}) = \tau$. Using (a), we can find $\lambda_0 \in Y_T$ such that $\tilde{\tau} \in \lambda_0(\mathbf{k}^*)$; moreover, by (b), we can assume that, setting $\lambda_0(t) = (\lambda_1(t), \lambda'(t))$ with $\lambda_1 \in Y_{\mathbf{k}^*}$, $\lambda' \in Y_{\mathcal{Z}_M^0}$, we have $\lambda_1(t) = t^f$ for all t where $f \in \mathbf{Z}_{>0}$, $f \notin p\mathbf{Z}$. Let $\lambda = d\lambda_0 : \mathbf{k}^* \to \iota(k^*)\mathcal{Z}_M^0$. We have $\lambda(t) = \iota(\lambda_1(t))\lambda'(t) = \iota(t^f)\lambda'(t)$ for $t \in \mathbf{k}^*$. Since $d(\tilde{\tau}) = \tau$ and $\tilde{\tau} \in \lambda_0(\mathbf{k}^*)$, we have $\tau \in \lambda(\mathbf{k}^*)$. This proves (c).

3.3. We now assume that τ as in 3.1(b) is given; let λ, λ', f be as in 3.2(c). We assume also that 3.1(a) holds. We can find $c \in \mathbf{k}^*$ of finite order such that $\lambda(c) = \tau$. (If $\tau \neq 1$, then λ is not identically 1 so it has finite kernel and any $c \in \lambda^{-1}(\tau)$ has finite order; if $\tau = 1$ we can take $c = 1$.)

Since $\lambda(\mathbf{k}^*) \subset M_0$ and $M_0 \subset G_{\underline{0}}$ (as a consequence of our assumption 3.1(a)), we can view λ as an element of $Y_{G_{\underline{0}}}$ hence ${}_k^\lambda\mathfrak{g}_i$ is defined for any $k \in \mathbf{Z}, i \in \mathbf{Z}/m$. Since $\lambda(\mathbf{k}^*) \subset M$, we can view λ as an element of Y_M hence ${}_k^\lambda\mathfrak{m}$ is defined for any $k \in \mathbf{Z}$.

For $t \in \mathbf{k}^*$, $k \in \mathbf{Z}$ such that $2k/\eta \in \mathbf{Z}$ and $x \in \mathfrak{m}_k$ we have $\mathrm{Ad}(\lambda(t))x = \mathrm{Ad}(\iota(t^f))\,\mathrm{Ad}(\lambda'(t))x = \mathrm{Ad}(\iota(t^f))x = t^{2kf/\eta}x$ (we use that $\lambda'(t) \in \mathcal{Z}_M^0$). Thus $\mathfrak{m}_k \subset {}_{2kf/\eta}^{\lambda}\mathfrak{m}$. Recall also that $\mathfrak{m}_k \neq 0$ implies $k/\eta \in \mathbf{Z}$; see 1.2(e). Since the subspaces $\mathfrak{m}_k$ form a direct sum decomposition of $\mathfrak{m}$ and the subspaces ${}_j^\lambda\mathfrak{m}$ form a direct sum decomposition of $\mathfrak{m}$, it follows that:

(a) $$\mathfrak{m}_k = {}^{\lambda}_{2kf/\eta}\mathfrak{m} \text{ for any } k \in \eta\mathbf{Z} \quad \text{and}$$
$${}^{\lambda}_{j}\mathfrak{m} = 0 \text{ unless } j = 2kf/\eta \quad \text{for some } k \in \eta\mathbf{Z}.$$

For $k \in \mathbf{Z}, i \in \mathbf{Z}/m$ and $x \in {}^{\lambda}_{k}\mathfrak{g}_i$ we have

$$\mathrm{Ad}(\tau)\theta(x) = \mathrm{Ad}(\lambda(c))\theta(x) = \zeta^i \mathrm{Ad}(\lambda(c))x = \zeta^i c^k x.$$

Since $\mathfrak{m} = \{x \in \mathfrak{g}; \mathrm{Ad}(\tau)(\theta(x)) = x\}$, we see that:

(b) $$\mathfrak{m} = \oplus_{j\in\mathbf{Z}, i\in\mathbf{Z}/m; \zeta^i c^j=1}({}^{\lambda}_{j}\mathfrak{g}_i).$$

If ${}^{\lambda}_{j}\mathfrak{g}_i$ is nonzero and contained in $\mathfrak{m}$ then ${}^{\lambda}_{j}\mathfrak{m}$ is nonzero hence by (a) we have $j = 2fk/\eta$ for some $k \in \mathbf{Z}$ and $\mathfrak{m}_k$ is a nonzero subspace of $\mathfrak{g}_i$; thus, by 3.1(a), we have $i = \underline{k}$ and $2k/\eta \in \mathbf{Z}$. Thus we can rewrite (b) as follows:

$$\mathfrak{m} = \oplus_{k\in\eta\mathbf{Z}; \zeta^k c^{2fk/\eta}=1}({}^{\lambda}_{2fk/\eta}\mathfrak{g}_{\underline{k}}),$$

that is,

(c) $$\mathfrak{m} = \oplus_{k\in\eta\mathbf{Z}; (\zeta^\eta c^{2f})^{k/\eta}=1}({}^{\lambda}_{2fk/\eta}\mathfrak{g}_{\underline{k}}).$$

Assume now that $\mathfrak{m}_\eta \neq 0$. Using (a) we have $\mathfrak{m}_\eta = {}^{\lambda}_{2f}\mathfrak{m} \neq 0$. By 3.1(a) we have $\mathfrak{m}_\eta \subset \mathfrak{g}_\delta$. It follows that $\mathfrak{m}$ has nonzero intersection with ${}^{\lambda}_{2f}\mathfrak{g}_\delta$. Now $\mathrm{Ad}(\tau)\theta$ acts on ${}^{\lambda}_{2f}\mathfrak{g}_\delta$ as multiplication by $\zeta^\eta c^{2f}$ and it acts on $\mathfrak{m}$ as the identity. It follows that $\zeta^\eta c^{2f} = 1$. Thus (c) can be rewritten as:

(d) $$\mathfrak{m} = \oplus_{k\in\eta\mathbf{Z}}({}^{\lambda}_{2fk/\eta}\mathfrak{g}_{\underline{k}}).$$

Next we assume that $\mathfrak{m}_\eta = 0$. By the definition of ι (see 3.1) this implies that ι is identically 1 hence $\mathfrak{m} = \mathfrak{m}_0$. From (a) we see that $\mathfrak{m} = {}^{\lambda}_{0}\mathfrak{m}$, hence in (c) all summands corresponding to $k \neq 0$ are zero. Thus (d) remains true in this case. We see also that

$$\mathfrak{m}_k = {}^{|\eta|\lambda}_{2fk\epsilon}\mathfrak{g}_{\underline{k}}$$

for all $k \in \mathbf{Z}$. Setting $\mu = |\eta|\lambda/(2f)$ we see that $\mathfrak{m}_*$ is a splitting of the p-regular ϵ-spiral ${}^{\epsilon}\mathfrak{p}_*^{\frac{1}{2f}|\eta|\lambda}$. We see that if $\dot{\xi}$ is admissible then it satisfies 3.1(c).

3.4. Assume now that $\dot{\xi}$ satisfies 3.1(c). Thus $\mathfrak{m}_*$ is a splitting of an ϵ-spiral $\mathfrak{p}_* = {}^{\epsilon}\mathfrak{p}_*^{\mu}$ where μ is p-regular. Applying the conjugacy result 2.7(a) to the two splittings $\mathfrak{m}_*, {}^{\epsilon}\tilde{\mathfrak{l}}_*^{\mu}$ we see that there exists a p-regular μ' such that $\mathfrak{p}_* = {}^{\epsilon}\mathfrak{p}_*^{\mu'}$, $\mathfrak{m}_* = {}^{\epsilon}\tilde{\mathfrak{l}}_*^{\mu'}$. Thus we can find $\lambda \in Y_{G_{\underline{0}}}$, $r \in \mathbf{Z}_{>0}$ such that $r \notin p\mathbf{Z}$ and

$$\mathfrak{m}_N = {}^{\lambda}_{rN\epsilon}\mathfrak{g}_{\underline{N}}$$

for any $N \in \mathbf{Z}$. In particular, 3.1(a) holds. We now show that 3.1(b) holds. From 2.6(c) we see that $M = G^{\mathrm{Ad}(\lambda(\zeta')^{-1})\vartheta}$ for some root of unity $\zeta' \in \mathbf{k}^*$. Let $\tau = \lambda(\zeta')^{-1}$. It remains to show that $\lambda(\zeta')^{-1} \in \iota(\mathbf{k}^*)\mathcal{Z}_M^0$. More generally, we show that $\lambda(t) \in \iota(\mathbf{k}^*)\mathcal{Z}_M^0$ for any $t \in \mathbf{k}^*$. Now λ can be viewed as an element of Y_M hence ${}^{\lambda}_{k}\mathfrak{m}$ is well-defined for any $k \in \mathbf{Z}$ and we have ${}^{\lambda}_{rN\epsilon}\mathfrak{m} = \mathfrak{m}_N$ for any $N \in \mathbf{Z}$. Recall that for $N \in \mathbf{Z}$ we have $\mathfrak{m}_N = {}^{\iota}_{2N/\eta}\mathfrak{m}$ if $N/\eta \in \mathbf{Z}$ and $\mathfrak{m}_N = 0$ if $N/\eta \notin \mathbf{Z}$. We see that for any $N \in \eta\mathbf{Z}$ and any $t \in \mathbf{k}^*$, $\mathrm{Ad}(\lambda(t))$ acts on $\mathfrak{m}_N$ as $t^{rN\epsilon}$ while $\mathrm{Ad}(\iota(t^{|\eta|}))$ acts on $\mathfrak{m}_N$ as $t^{2N\epsilon}$. Hence $\mathrm{Ad}(\lambda(t)^2\iota(t)^{-r|\eta|})$ acts on $\mathfrak{m}_N$ as 1. Since $\mathfrak{m}$ is the sum of the subspaces $\mathfrak{m}_N$, we see that $\mathrm{Ad}(\lambda(t)^2\iota(t)^{-r|\eta|})$ acts on $\mathfrak{m}$ as 1. It follows that $\lambda(t)^2\iota(t)^{-r|\eta|} \in \mathcal{Z}_M$. Since $t \mapsto \lambda(t)^2\iota(t)^{-r|\eta|}$ is a homomorphism of the connected group $\mathbf{k}^*$ into $\mathcal{Z}_M$, its image must be contained in $\mathcal{Z}_M^0$. Thus, for any $t \in \mathbf{k}^*$ we have $\lambda(t)^2\iota(t)^{-r|\eta|} \in \mathcal{Z}_M^0$ hence $\lambda(t^2) \in \iota(\mathbf{k}^*)\mathcal{Z}_M^0$. Since any $t' \in \mathbf{k}^*$ is a square, it follows that $\lambda(t') \in \iota(\mathbf{k}^*)\mathcal{Z}_M^0$ for any $t' \in \mathbf{k}^*$. We see that, if $\dot{\xi}$ satisfies 3.1(c), then $\dot{\xi}$ is admissible. This completes the proof of 3.1(d).

3.5. **The map** $\Psi : \mathcal{I}(\mathfrak{g}_\delta) \to \underline{\mathfrak{T}}_\eta$**.** Let $\mathcal{I}(\mathfrak{g}_\delta)$ be the set of pairs $(\mathcal{O}, \mathcal{L})$ where $\mathcal{O}$ is a $G_{\underline{0}}$-orbit on $\mathfrak{g}_\delta^{nil}$ and $\mathcal{L}$ is an irreducible $G_{\underline{0}}$-equivariant local system on $\mathcal{O}$ defined up to isomorphism. Since $G_{\underline{0}}$ acts on $\mathfrak{g}_\delta^{nil}$ with finitely many orbits, see [Vi], the set $\mathcal{I}(\mathfrak{g}_\delta)$ is finite.

Let $\mathfrak{T}_\eta$ be the set of all $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{T}'_\eta$ which are admissible (see 3.1) or equivalently (see 3.1(d)) are such that $\mathfrak{m}_*$ is a splitting of some p-regular ϵ-spiral. The group $G_{\underline{0}}$ acts in an obvious way by conjugation on $\mathfrak{T}_\eta$; we denote by $\underline{\mathfrak{T}}_\eta$ the set of orbits, which is a finite set. We will define a map $\Psi : \mathcal{I}(\mathfrak{g}_\delta) \to \underline{\mathfrak{T}}_\eta$. Let $(\mathcal{O}, \mathcal{L}) \in \mathcal{I}(\mathfrak{g}_\delta)$. Choose $x \in \mathcal{O}$ and $\phi \in J_\delta(x)$; define $\mathfrak{u}_*^x$, $\tilde{\mathfrak{l}}_*^\phi$, $\tilde{\mathfrak{l}}^\phi, \tilde{L}^\phi, \tilde{L}_0^\phi$ as in 2.9. Recall that $\tilde{L}^\phi \in \mathfrak{G}$. We have $x \in \overset{\circ}{\tilde{\mathfrak{l}}}{}_\eta^\phi$ (see 2.9(b)). By 2.9(c), $\mathcal{L}_1 := \mathcal{L}|_{\overset{\circ}{\tilde{\mathfrak{l}}}{}_\eta^\phi}$ is an irreducible $\tilde{L}_0^\phi$-equivariant local system on $\overset{\circ}{\tilde{\mathfrak{l}}}{}_\eta^\phi$. Let A be the simple $\tilde{L}_0^\phi$-equivariant perverse sheaf on $\tilde{\mathfrak{l}}_\eta^\phi$ whose restriction to $\overset{\circ}{\tilde{\mathfrak{l}}}{}_\eta^\phi$ is $\mathcal{L}_1[\dim \tilde{\mathfrak{l}}_\eta^\phi]$. The map 1.5(b) associates to A an element $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$ of $\mathfrak{M}_\eta(\tilde{L}^\phi)$ well defined up to conjugation by $\tilde{L}_0^\phi$. Using 1.6(a) we can find a parabolic subalgebra $\mathfrak{q}$ of $\tilde{\mathfrak{l}}^\phi$ compatible with the $\mathbf{Z}$-grading of $\tilde{\mathfrak{l}}^\phi$ and such that $\mathfrak{m}$ is a Levi subalgebra of $\mathfrak{q}$. Setting $\mathfrak{p}'_N = \mathfrak{u}_N^\phi + \mathfrak{q}_N$ for any $N \in \mathbf{Z}$, we see from 2.8(a) that $\mathfrak{p}'_*$ is a p-regular ϵ-spiral and from 2.8(b) that $\mathfrak{m}_*$ is a splitting of $\mathfrak{p}'_*$. We see that $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{T}_\eta$.

We now show that the $G_{\underline{0}}$-orbit of $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$ is independent of the choices made. First, if x, ϕ are already chosen, then the $\tilde{L}_0^\phi$-orbit of $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$ is well defined hence the $G_{\underline{0}}$-orbit of $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$ is well defined (since $\tilde{L}_0^\phi \subset G_{\underline{0}}$). The independence of the choice of ϕ (when x is given) follows from the homogeneity of $J_\delta(x)$ under the group $U \cap G_{\underline{0}}$ in 2.3(b). Finally, the independence of the choice of x follows from the homogeneity of $\mathcal{O}$ under the group $G_{\underline{0}}$. Thus,

$$(\mathcal{O}, \mathcal{L}) \mapsto (G_{\underline{0}} - \text{orbit of } (M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}))$$

is a well-defined map $\Psi : \mathcal{I}(\mathfrak{g}_\delta) \to \underline{\mathfrak{T}}_\eta$.

3.6. Let $\dot{\xi} = (M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{T}_\eta$. Let $\mathcal{O}_{\dot{\xi}}$ be the unique $G_{\underline{0}}$-orbit in $\mathfrak{g}_\delta^{nil}$ that contains $\overset{\circ}{\mathfrak{m}}_\eta$. Let $\dot{\xi}' = (M', M'_0, \mathfrak{m}', \mathfrak{m}'_*, \tilde{C}') \in \mathfrak{T}_\eta$. We show:

(a) *If* $\mathcal{O}_{\dot{\xi}} = \mathcal{O}_{\dot{\xi}'}$, *then there exists* $g \in G_{\underline{0}}$ *such that* $\mathrm{Ad}(g)$ *carries* $(M, M_0, \mathfrak{m}, \mathfrak{m}_*)$ *to* $(M', M'_0, \mathfrak{m}', \mathfrak{m}'_*)$.

By [L4, 3.3], we can find $\phi = (e, h, f) \in J^M$, $\phi' = (e', h', f') \in J^{M'}$ such that:

(b) $e \in \overset{\circ}{\mathfrak{m}}_\eta$, $h \in \mathfrak{m}_0$, $f \in \mathfrak{m}_{-\eta}$, $e' \in \overset{\circ}{\mathfrak{m}}{}'_\eta$, $h' \in \mathfrak{m}'_0$, $f' \in \mathfrak{m}'_{-\eta}$.

We set $\iota = \iota_\phi \in Y_M$, $\iota' = \iota_{\phi'} \in Y_{M'}$. By 1.2(a),(c),(e), we have

(c) $$\mathfrak{m}_k = {}_{2k/\eta}^{\iota}\mathfrak{m}, \quad \mathfrak{m}'_k = {}_{2k/\eta}^{\iota'}\mathfrak{m}' \text{ if } k \in \eta\mathbf{Z}, \mathfrak{m}_k = \mathfrak{m}'_k = 0 \text{ if } k \in \mathbf{Z} - \eta\mathbf{Z}.$$

By assumption, we have $e' = \mathrm{Ad}(g_1)e$ for some $g_1 \in G_{\underline{0}}$. Replacing the system $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}, \phi)$ by its image under $\mathrm{Ad}(g_1)$, we see that we can assume that $e = e'$. Using 3.1(a) for $\dot{\xi}$ and $\dot{\xi}'$, we can view ϕ, ϕ' as elements of J_δ^G with the same first component. By 2.3(b), we can find $g_2 \in G_{\underline{0}}$ such that $\mathrm{Ad}(g_2)$ carries ϕ to ϕ'. Replacing $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}, \phi)$ by its image under $\mathrm{Ad}(g_1)$, we see that we can assume that $\phi = \phi'$ as elements of J^G. It follows that $\iota = \iota'$ as elements of Y_G.

Let

$$G_\phi = \{g \in G; \mathrm{Ad}(g)(e) = e, \mathrm{Ad}(g)(h) = h, \mathrm{Ad}(g)(f) = f\}.$$

Since e, h, f are contained in $\mathfrak{m}$ we have $\mathcal{Z}_M \subset G_\phi$. Similarly, since e, h, f are contained in $\mathfrak{m}'$, we have $\mathcal{Z}_{M'} \subset G_\phi$. We have also $\mathcal{Z}_M^0 \subset G_{\underline{0}}$ (since the center of $\mathfrak{m}$ is contained in $\mathfrak{m}_0 \subset \mathfrak{g}_{\underline{0}}$); similarly we have $\mathcal{Z}_{M'}^0 \subset G_{\underline{0}}$. Thus, $\mathcal{Z}_M^0$ and $\mathcal{Z}_{M'}^0$ are tori in $(G_\phi \cap G_{\underline{0}})^0$. We show that $\mathcal{Z}_M^0$ is a maximal torus of $(G_\phi \cap G_{\underline{0}})^0$. Indeed, assume that S is a torus of $(G_\phi \cap G_{\underline{0}})^0$ that contains $\mathcal{Z}_M^0$. Since $S \subset G_\phi$, for any $s \in S$ we have $\mathrm{Ad}(s)h = h$ hence $s\iota(t) = \iota(t)s$, that is, $\mathrm{Ad}(\iota(t))s = s$ for $t \in \mathbf{k}^*$. Since S contains $\mathcal{Z}_M^0$, for any $s \in S, z \in \mathcal{Z}_M^0$ we have $\mathrm{Ad}(z)s = s$. Since $S \subset G_{\underline{0}}$ we have $\vartheta(s) = s$ for any $s \in S$. We see that $\mathrm{Ad}(\iota(t))\,\mathrm{Ad}(z)\vartheta(s) = s$ for any $t \in \mathbf{k}^*$, $z \in \mathcal{Z}_M^0$, $s \in S$. We can find $\tau \in \iota(\mathbf{k}^*)\mathcal{Z}_M^0$ such that $M = G^{\mathrm{Ad}(\tau)\vartheta}$. We have seen that $\mathrm{Ad}(\tau)\vartheta(s) = s$ for $s \in S$. Thus $S \subset M$. Since $S \subset G_\phi$, we have

$$S \subset M_\phi := \{g \in M; \mathrm{Ad}(g)(e) = e, \mathrm{Ad}(g)(h) = h, \mathrm{Ad}(g)(f) = f\},$$

hence $S \subset M_\phi^0$. Since e is a distinguished nilpotent element of $\mathfrak{m}$, we have $M_\phi^0 = \mathcal{Z}_M^0$. Thus we have $S \subset \mathcal{Z}_M^0$. By assumption, we have $\mathcal{Z}_M^0 \subset S$, hence $\mathcal{Z}_M^0 = S$. Thus $\mathcal{Z}_M^0$ is indeed a maximal torus of $(G_\phi \cap G_{\underline{0}})^0$, as claimed. Similarly we see that $\mathcal{Z}_{M'}^0$ is a maximal torus of $(G_\phi \cap G_{\underline{0}})^0$. Since any two maximal tori of $(G_\phi \cap G_{\underline{0}})^0$ are conjugate, we can find g_3 in $(G_\phi \cap G_{\underline{0}})^0$ such that $\mathrm{Ad}(g_3)$ carries $\mathcal{Z}_M^0$ to $\mathcal{Z}_{M'}^0$. (It also carries ϕ to ϕ.)

Replacing $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}, \phi)$ by its image under $\mathrm{Ad}(g_3)$, we see that we can assume that $\mathcal{Z}_M^0 = \mathcal{Z}_{M'}^0$ and $\phi = \phi'$.

Assume now that $e = 0$ so that $e' = 0$. By the definition of $\iota = \iota'$ we see that $\iota = \iota'$ is identically 1 hence $\mathfrak{m} = \mathfrak{m}_0$, $\mathfrak{m}' = \mathfrak{m}'_0$ and $G_\phi = G$. Since $e = 0$ is distinguished in $\mathfrak{m}$ it follows that M is a torus. Hence $M = \mathcal{Z}_M^0$. Similarly $M' = \mathcal{Z}_{M'}^0$. Since $\mathcal{Z}_M^0 = \mathcal{Z}_{M'}^0$ it follows that $M = M'$. We see that (a) holds in this case.

In the remainder of the proof we assume that $e \neq 0$ hence $e' \neq 0$. Recall that $M = G^{\mathrm{Ad}(\iota(t))\,\mathrm{Ad}(z)\vartheta}$, $M' = G^{\mathrm{Ad}(\iota(t'))\,\mathrm{Ad}(z')\vartheta}$, for some t, t' in $\mathbf{k}^*$ and some z, z' in $\mathcal{Z}_M^0 = \mathcal{Z}_{M'}^0$. Since $e \in \mathfrak{m}_\eta$, we have $\mathrm{Ad}(\iota(t))\,\mathrm{Ad}(z)\theta(e) = e$; since $\mathrm{Ad}(z)$ acts as 1 on $\mathfrak{m}$, we deduce that $t^2\zeta^\eta e = e$ and since $e \neq 0$, we see that $t^2 = \zeta^{-\eta}$. Similarly, since $e \in \mathfrak{m}'_\eta$ we have $\mathrm{Ad}(\iota(t'))\,\mathrm{Ad}(z')\theta(e) = e$ and $t'^2 = \zeta^{-\eta}$.

We show that for any $k \in \mathbf{Z}$ we have $\mathfrak{m}_k \subset \mathfrak{m}'$. By 1.2(e) we can assume that $k \in \eta\mathbf{Z}$. Let $x \in \mathfrak{m}_k$. We must show that $\mathrm{Ad}(\iota(t'))\,\mathrm{Ad}(z')\theta(x) = x$. Since $\mathrm{Ad}(z')$ acts by 1 on $\mathfrak{m}$, it is enough to show that $\zeta^k t'^{2k/\eta} x = x$ or that $(\zeta^\eta t'^2)^{k/\eta} x = x$. This follows from $t'^2 = \zeta^{-\eta}$.

Thus we have $\mathfrak{m}_k \subset \mathfrak{m}'$. Since this holds for any $k \in \mathbf{Z}$, we deduce that $\mathfrak{m} \subset \mathfrak{m}'$. Interchanging the roles of $\mathfrak{m}, \mathfrak{m}'$ we see that $\mathfrak{m}' \subset \mathfrak{m}$ hence $\mathfrak{m} = \mathfrak{m}'$. This implies that $M = M'$. Since $\iota = \iota'$, we see from (c) that $\mathfrak{m}_* = \mathfrak{m}'_*$. From $\mathfrak{m}_0 = \mathfrak{m}'_0$ we deduce that $M_0 = M'_0$. This completes the proof of (a).

The following result can be extracted from the proof of (a).

(d) *If $\mathfrak{m}_\eta = 0$ (so that $e = 0$), then $\mathfrak{m} = \mathfrak{m}_0$ is a Cartan subalgebra of $\mathfrak{g}_{\underline{0}}$.*

3.7. Let $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{T}_\eta$. Let $x \in \overset{\circ}{\mathfrak{m}}_\eta$. We choose $\phi = (e, h, f) \in J^M$ such that $e = x, h \in \mathfrak{m}_0, f \in \mathfrak{m}_{-\eta}$ (see [L4, 3.3]). We can view x as an element of $\mathfrak{g}_\delta^{nil}$ and ϕ as an element of $J_\delta(x)$. We define $\tilde{\mathfrak{l}}_* = \tilde{\mathfrak{l}}_*^\phi$ as in 2.9. Recall that for $N \in \mathbf{Z}$ we have:

$$\tilde{\mathfrak{l}}_N = {}_{2N/\eta}^{\iota}\mathfrak{g}_{\underline{N}} \text{ if } 2N/\eta \in \mathbf{Z}, \quad \tilde{\mathfrak{l}}_N = 0 \text{ if } 2N/\eta \notin \mathbf{Z},$$

where $\iota = \iota_\phi \in Y_G$. Let $\tilde{\mathfrak{l}} = \oplus_N \tilde{\mathfrak{l}}_N \subset \mathfrak{g}$ and let $\tilde{L} = e^{\tilde{\mathfrak{l}}} \subset G$. We show:

(a) $\mathfrak{m}$ *is a Levi subalgebra of a parabolic subalgebra of* $\tilde{\mathfrak{l}}$ *which is compatible with the* $\mathbf{Z}$*-grading of* $\tilde{\mathfrak{l}}$*.*

We shall prove (a) without the statement of compatibility with the $\mathbf{Z}$-grading; then the full statement of (a) would follow from 1.6(a).

Assume first that $x = 0$. Then $h = 0$ hence ι is the constant map with image 1. It follows that $\tilde{\mathfrak{l}} = \tilde{\mathfrak{l}}_0 = \mathfrak{g}_{\underline{0}}$ and $\mathfrak{m} = \mathfrak{m}_0$; moreover: by 3.6(d), $\mathfrak{m}$ is a Cartan subalgebra of $\mathfrak{g}_{\underline{0}}$. Hence in this case (a) is immediate. In the rest of the proof we assume that $x \neq 0$.

Since $\overset{\circ}{\mathfrak{m}}_\eta$ carries a cuspidal local system, for any $N \in \mathbf{Z}$ such that $2N/\eta \in \mathbf{Z}$ we have $\mathfrak{m}_N = {}_{2N/\eta}^{\iota}\mathfrak{m}$. Since $\mathfrak{m}_N \subset \mathfrak{g}_{\underline{N}}$, we have $\mathfrak{m}_N \subset {}_{2N/\eta}^{\iota}\mathfrak{g}_{\underline{N}}$ hence $\mathfrak{m}_N \subset \tilde{\mathfrak{l}}_N$. Taking sum over all $N \in \mathbf{Z}$ such that $2N/\eta \in \mathbf{Z}$, we get $\mathfrak{m} \subset \tilde{\mathfrak{l}}$. We can find $t_0 \in \mathbf{k}^*$, $z \in \mathcal{Z}_M^0$, both of finite order, such that $\mathfrak{m} = \{y \in \mathfrak{g}; \mathrm{Ad}(\iota(t_0)) \mathrm{Ad}(z)\theta(y) = y\}$. Note that $\tilde{\mathfrak{l}}_* = {}^{\dot\eta}\tilde{\mathfrak{l}}_*^{(|\eta|/2)\iota}$

By 2.6(c), we can find $\zeta' \in \mathbf{k}^*$ such that $\tilde{\mathfrak{l}} = \{y \in \mathfrak{g}; \mathrm{Ad}(\iota(\zeta')^{-1})\theta(y)) = y\}$. Since $\mathfrak{m} \subset \tilde{\mathfrak{l}}$, we have:

(b) $\mathfrak{m} = \{y \in \tilde{\mathfrak{l}}; \mathrm{Ad}(\iota(t_0)) \mathrm{Ad}(z)\theta(y) = y\} = \{y \in \tilde{\mathfrak{l}}; \mathrm{Ad}(\iota(t_0)) \mathrm{Ad}(z) \mathrm{Ad}(\iota(\zeta'))y = y\}.$

(Note that 2.6(c) is applicable since $\tilde{\mathfrak{l}}_* = {}^{\dot\eta}\tilde{\mathfrak{l}}_*^{(|\eta|/2)\iota}$.)

Since $x \in \mathfrak{m}_\eta \subset {}_2^\iota\mathfrak{g}$, we have $\mathrm{Ad}(\iota(t))x = t^2 x$ for any t. Taking $t = t_0^{-1}$ or $t = \zeta'$ we see that $t_0^{-2}x = \mathrm{Ad}(\iota(t_0))^{-1}x$ and $\zeta'^2 x = \mathrm{Ad}(\iota(\zeta'))x$. Since $x \in \mathfrak{m}$ and $x \in \tilde{\mathfrak{l}}$ we have $\mathrm{Ad}(\iota(t_0))^{-1}x = \theta(x)$ and $\mathrm{Ad}(\iota(\zeta'))x = \theta(x)$. It follows that $t_0^{-2}x = \zeta'^2 x$ so that (since $x \neq 0$) we have $t_0^{-2} = \zeta'^2$.

If $N \in \mathbf{Z}, 2N/\eta \in \mathbf{Z}$ and $y \in \tilde{\mathfrak{l}}_N$, we have $\mathrm{Ad}(\iota(t_0\zeta'))y = (t_0\zeta')^{2N/\eta}y = y$. Since $\tilde{\mathfrak{l}} = \oplus_N \tilde{\mathfrak{l}}_N$ we have $\mathrm{Ad}(\iota(t_0\zeta'))y = y$ for all $y \in \tilde{\mathfrak{l}}$. Hence (b) implies:

(c) $\mathfrak{m} = \{y \in \tilde{\mathfrak{l}}; \mathrm{Ad}(z)y = y\}$.

It remains to show that (c) implies (a). Since z is of finite order and $z \in \mathcal{Z}_M^0$, we can find $\lambda \in Y_{\mathcal{Z}_M^0}$ such that $z = \lambda(t_1)$ for some $t_1 \in \mathbf{k}^*$. (See 3.2(a).)

Let $\mathfrak{m}' = \{y \in \tilde{\mathfrak{l}}; \mathrm{Ad}(\lambda(t))y = y \quad \forall t \in \mathbf{k}^*\}$. Note that $\mathfrak{m}'$ is a Levi subalgebra of a parabolic subalgebra $\mathfrak{q}$ of $\tilde{\mathfrak{l}}$. Since $\lambda(\mathbf{k}^*) \subset \mathcal{Z}_M^0$ we see that $\mathrm{Ad}(\lambda(t))$ acts as 1 on $\mathfrak{m}$ for any t hence $\mathfrak{m} \subset \mathfrak{m}'$. Now $\mathrm{Ad}(\lambda(t_1))$ acts as 1 on $\mathfrak{m}'$. Since $\mathfrak{m} = \{y \in \tilde{\mathfrak{l}}; \mathrm{Ad}(\lambda(t_1))y = y\}$ it follows that $\mathfrak{m}' = \mathfrak{m}$. Thus (a) holds.

3.8. **Primitive pairs.** Let $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{T}_\eta$. Let $x \in \overset{\circ}{\mathfrak{m}}_\eta$. We can view x as an element of $\mathfrak{g}_\delta^{nil}$. We set $M_0(x) = M_0 \cap G(x)$, $G_{\underline{0}}(x) = G_{\underline{0}} \cap G(x)$. We show:

(a) *The inclusion* $M_0(x) \to G_{\underline{0}}(x)$ *induces an isomorphism on the groups of components.*

Let $\phi \in J^M$, $\tilde{\mathfrak{l}}, \tilde{\mathfrak{l}}_*, \tilde{L}$ be as in 3.7. Let $\tilde{L}_0 = e^{\tilde{\mathfrak{l}}_0} \subset \tilde{L}$. We have $x \in \overset{\circ}{\tilde{\mathfrak{l}}}_\eta$ (see [L4, 4.2(a)]). Let $\tilde{L}_0(x) = \tilde{L}_0 \cap G(x)$. To prove (a) it is enough to prove (i) and (ii) below.

(i) *The inclusion* $M_0(x) \to \tilde{L}_0(x)$ *induces an isomorphism on the groups of components.*

(ii) *The inclusion* $\tilde{L}_0(x) \to G_{\underline{0}}(x)$ *induces an isomorphism on the groups of components.*

Now (i) follows from [L4, 11.2] (we use 3.7(a)) and (ii) is a special case of 2.9(c). This proves (a).

Let $\mathcal{O}$ be the $G_{\underline{0}}$-orbit of x in $\mathfrak{g}_\delta^{nil}$. Let $\mathcal{L}'$ be the irreducible M_0-equivariant local system on $\mathring{\mathfrak{m}}_\eta$ such that $\tilde{C}|_{\mathring{\mathfrak{m}}_\eta} = \mathcal{L}'[\dim \mathfrak{m}_\eta]$. Let $\mathcal{L}$ be the irreducible $G_{\underline{0}}$-equivariant local system on $\mathcal{O}$ which corresponds to $\mathcal{L}'$ under (a). We say that $(\mathcal{O}, \mathcal{L}) \in \mathcal{I}(\mathfrak{g}_\delta)$ is the *primitive pair* corresponding to $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \in \mathfrak{T}_\eta$; it is clearly independent of the choice of x, ϕ (we use [L4, 3.3]).

Let $\mathcal{L}''$ be the irreducible $\tilde{L}_0$-equivariant local system on $\mathring{\tilde{\mathfrak{l}}}_\eta$ which corresponds to $\mathcal{L}'$ under (i). Let $\mathcal{L}''^\sharp \in \mathcal{D}(\tilde{\mathfrak{l}}_\eta)$ be as in 0.11. From 1.8(b) we see that:

(b) $\mathrm{ind}_{\mathfrak{q}_\eta}^{\tilde{\mathfrak{l}}_\eta}(\tilde{C})$ *is a nonzero direct sum of shifts of* $\mathcal{L}''^\sharp$.

Consider the map $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \mapsto (\mathcal{O}, \mathcal{L})$ (as above) from $\mathfrak{T}_\eta$ to $\mathcal{I}(\mathfrak{g}_\delta)$; the image of this map is denoted by $\mathcal{I}^{prim}(\mathfrak{g}_\delta)$. From 3.6(a) and (a) we see that:

(c) *This induces a bijection* $\omega : \underline{\mathfrak{T}}_\eta \xrightarrow{\sim} \mathcal{I}^{prim}(\mathfrak{g}_\delta)$.

Using the definitions and 1.8(b), we see that:

(d) *For* $\xi \in \underline{\mathfrak{T}}_\eta$ *we have* $\Psi(\omega(\xi)) = \xi$, *where* $\Psi : \mathcal{I}(\mathfrak{g}_\delta) \to \underline{\mathfrak{T}}_\eta$ *is as in* 3.5.

Combining (c) and (d), we have

(e) *the restriction of* Ψ *to* $\mathcal{I}^{prim}(\mathfrak{g}_\delta)$ *gives the inverse of* ω.

From (d) we get:

(f) *The map* $\Psi : \mathcal{I}(\mathfrak{g}_\delta) \to \underline{\mathfrak{T}}_\eta$ *is surjective.*

Another proof of (f) is given in 7.3.

3.9. Now let $\eta_1 \in \mathbf{Z} - \{0\}$ be such that $\underline{\eta_1} = \delta$. We define a bijection $\mathfrak{T}'_\eta \xrightarrow{\sim} \mathfrak{T}'_{\eta_1}$ by $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}) \mapsto (M, M_0, \mathfrak{m}, \mathfrak{m}_{(*)}, \tilde{C})$ where $\mathfrak{m}_{(*)}$ is the new $\mathbf{Z}$-grading on $\mathfrak{m}_*$ whose k-component $\mathfrak{m}_{(k)}$ is equal to $\mathfrak{m}_{k\eta/\eta_1}$ for any $k \in \eta_1\mathbf{Z}$ and is equal to 0 for any $k \in \mathbf{Z} - \eta_1\mathbf{Z}$. (This is well defined since $\mathfrak{m}_{k'} = 0$ for any $k' \in \mathbf{Z} - \eta\mathbf{Z}$; see 1.2(e).) Here we regard $\tilde{C}$ as a simple perverse sheaf on $\mathfrak{m}_\eta = \mathfrak{m}_{(\eta_1)}$. This restricts to a bijection $\mathfrak{T}_\eta \xrightarrow{\sim} \mathfrak{T}_{\eta_1}$, which induces a bijection $\underline{\mathfrak{T}}_\eta \xrightarrow{\sim} \underline{\mathfrak{T}}_{\eta_1}$. This allows us to identify canonically all the sets $\mathfrak{T}_{\eta_1}$ (for various $\eta_1 \in \mathbf{Z} - \{0\}$ such that $\underline{\eta_1} = \delta$) with a single set $\mathfrak{T}_\delta$ and also all the sets $\underline{\mathfrak{T}}_{\eta_1}$ (for various $\eta_1 \in \mathbf{Z} - \{0\}$ such that $\underline{\eta_1} = \delta$) with a single set $\underline{\mathfrak{T}}_\delta$. Here $\mathfrak{T}_\delta$, $\underline{\mathfrak{T}}_\delta$ are defined purely in terms of δ (independently of the choice of η).

An examination of the construction of the map $\Psi = \Psi_\eta : \mathcal{I}(\mathfrak{g}_\delta) \to \underline{\mathfrak{T}}_\eta$ (see 3.5) shows that the bijection $\underline{\mathfrak{T}}_\eta \xrightarrow{\sim} \underline{\mathfrak{T}}_{\eta_1}$ intertwines Ψ_η and Ψ_{η_1}. Therefore we have a well-defined map $\Psi : \mathcal{I}(\mathfrak{g}_\delta) \to \underline{\mathfrak{T}}_\delta$.

4. Spiral induction

In this section we introduce the key tool in studying the block decomposition for $\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_\delta^{nil})$, namely the spiral induction. This is the analogue in the $\mathbf{Z}/m$-graded setting of the parabolic induction in the ungraded or $\mathbf{Z}$-graded setting.

4.1. **Definition of spiral induction.** In addition to $\eta \in \mathbf{Z} - \{0\}$ which has been fixed in 2.9, in this section we fix $\epsilon \in \{1, -1\}$. We denote by $\mathfrak{P}^\epsilon$ the set of all data of the form:

(a) $$(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*),$$

where $\mathfrak{p}_*$ is an ϵ-spiral and $L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*$ are associated to $\mathfrak{p}_*$ as in 2.5. Let

$$(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*) \in \mathfrak{P}^\epsilon.$$

Let $\pi : \mathfrak{p}_\eta \to \mathfrak{l}_\eta$ be the obvious projection. We have a diagram:

(b) $$\mathfrak{l}_\eta \xleftarrow{a} G_{\underline{0}} \times \mathfrak{p}_\eta \xrightarrow{b} E' \xrightarrow{c} \mathfrak{g}_\delta,$$

where $E' = \{(gP_0, z) \in G_{\underline{0}}/P_0 \times \mathfrak{g}_\delta; \mathrm{Ad}(g^{-1})z \in \mathfrak{p}_\eta\}$, $a(g, z) = \pi(\mathrm{Ad}(g^{-1})z)$, $b(g, z) = (gP_0, z)$, $c(gP_0, z) = z$. Here a is smooth with connected fibers, b is a principal P_0-bundle and c is proper. Now $\mathcal{Q}(\mathfrak{l}_\eta)$ is defined as in 1.2, with $H, \mathfrak{h}$ replaced by $L, \mathfrak{l}$. If $A \in \mathcal{Q}(\mathfrak{l}_\eta)$, then a^*A is a P_0-equivariant semisimple complex on $G_{\underline{0}} \times \mathfrak{p}_\eta$, hence there is a well-defined semisimple complex A_1 on E' such that $b^*A_1 = a^*A$. We can form the complex

$$^\epsilon\,\mathrm{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A) = c_! A_1.$$

Since c is proper, this is a semisimple, $G_{\underline{0}}$-equivariant complex on $\mathfrak{g}_\delta$.

If $\tilde{\mathfrak{l}}_*$ is a splitting of $\mathfrak{p}_*$, we will sometimes consider $^\epsilon\,\mathrm{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)$ with $A \in \mathcal{Q}(\tilde{\mathfrak{l}}_\eta)$ by identifying $\tilde{\mathfrak{l}}_\eta$ with $\mathfrak{l}_\eta$ in an obvious way and A with an object in $\mathcal{Q}(\mathfrak{l}_\eta)$.

For any $A \in \mathcal{Q}(\mathfrak{l}_\eta)$ we have

(c) $$D(^\epsilon\,\mathrm{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)) = {}^\epsilon\,\mathrm{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(D(A))[2e],$$

where e is the dimension of a fiber of a minus the dimension of a fiber of b, or equivalently

$$\begin{aligned} e &= \dim \mathfrak{g}_{\underline{0}} + \dim \mathfrak{p}_\eta - \dim \mathfrak{u}_0 - (\dim \mathfrak{p}_\eta - \dim \mathfrak{u}_\eta) - (\dim \mathfrak{p}_0 - \dim \mathfrak{u}_0) \\ &= \dim \mathfrak{u}_0 + \dim \mathfrak{u}_\eta. \end{aligned}$$

Hence, if for $A \in \mathcal{Q}(\mathfrak{l}_\eta)$ we set

$$^\epsilon\widetilde{\mathrm{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A) = {}^\epsilon\,\mathrm{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)[\dim \mathfrak{u}_0 + \dim \mathfrak{u}_\eta],$$

then

(d) $$D(^\epsilon\widetilde{\mathrm{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)) = {}^\epsilon\widetilde{\mathrm{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(D(A)).$$

4.2. **Transitivity.** We state a transitivity property of induction. In addition to the datum 4.1(a) we consider a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{l}$ such that $\mathfrak{q} = \oplus_{N \in \mathbf{Z}} \mathfrak{q}_N$ where $\mathfrak{q}_N = \mathfrak{q} \cap \mathfrak{l}_N$. For any $N \in \mathbf{Z}$ let $\hat{\mathfrak{p}}_N$ be the inverse image of $\mathfrak{q}_N$ under the obvious map $\mathfrak{p}_N \to \mathfrak{l}_N$. Then $\hat{\mathfrak{p}}_*$ is an ϵ-spiral; see 2.8(a). Let

$$(\hat{\mathfrak{p}}_*, \hat{L}, \hat{P}_0, \hat{\mathfrak{l}}, \hat{\mathfrak{l}}_*, \hat{\mathfrak{u}}_*) \in \mathfrak{P}^\epsilon$$

be the datum analogous to 4.1(a) defined by $\hat{\mathfrak{p}}_*$. Now $\mathcal{Q}(\hat{\mathfrak{l}}_\eta)$ is defined as in 1.2, with $H, \mathfrak{h}$ replaced by $\hat{L}, \hat{\mathfrak{l}}$. If $A \in \mathcal{Q}(\hat{\mathfrak{l}}_\eta)$, then $\mathrm{ind}_{\mathfrak{q}_\eta}^{\mathfrak{l}_\eta}(A) \in \mathcal{Q}(\mathfrak{l}_\eta)$ is defined as in 1.3 and we have canonically

(a) $$^\epsilon\,\mathrm{Ind}_{\hat{\mathfrak{p}}_\eta}^{\mathfrak{g}_\delta}(A) = {}^\epsilon\,\mathrm{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\mathrm{ind}_{\mathfrak{q}_\eta}^{\mathfrak{l}_\eta}(A)).$$

The proof is similar to that of [L2, 4.2]; it is omitted.

4.3. In the setup of 4.1, assume that $A \in \mathcal{Q}(\mathfrak{l}_\eta)$ is a cuspidal perverse sheaf (see 1.2). We have $A = \mathcal{L}^\sharp[\dim \mathfrak{l}_\eta]$ where $\mathcal{L}$ is an irreducible local system on $\mathring{\mathfrak{l}}_\eta$ and $\mathcal{L}^\sharp \in \mathcal{D}(\mathfrak{l}_\eta)$ is as in 0.11. In this case we can give an alternative description of ${}^\epsilon \mathrm{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\mathcal{L}^\sharp)$. Let P_0, $\pi : \mathfrak{p}_\eta \to \mathfrak{l}_\eta$ be as in 4.1. Let

$$\dot{\mathfrak{g}}_\delta = \{(gP_0, z) \in G_{\underline{0}}/P_0 \times \mathfrak{g}_\delta; \mathrm{Ad}(g^{-1})z \in \pi^{-1}(\mathring{\mathfrak{l}}_\eta)\}$$

be an open smooth irreducible subvariety of E' in 4.1. Let $\dot{\mathcal{L}}$ be the local system on $\dot{\mathfrak{g}}_\delta$ defined by $b'^*\dot{\mathcal{L}} = a'^*\mathcal{L}$, where

$$\mathring{\mathfrak{l}}_\eta \xleftarrow{a'} G_{\underline{0}} \times (\pi^{-1}(\mathring{\mathfrak{l}}_\eta)) \xrightarrow{b'} \dot{\mathfrak{g}}_\delta,$$

$a'(g, zx) = \pi(\mathrm{Ad}(g^{-1})z), b'(g, z) = (gP_0, z)$. Let $\dot{\mathcal{L}}^\sharp$ be the intersection cohomology complex of E' with coefficients in $\dot{\mathcal{L}}$. From the definitions we have $a^*\mathcal{L}^\sharp = b^*\dot{\mathcal{L}}^\sharp$ (a, b as in 4.1). We define $c' : \dot{\mathfrak{g}}_\delta \to \mathfrak{g}_\delta$ by $c'(g, z) = z$. We show:

(a) $${}^\epsilon \mathrm{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\mathcal{L}^\sharp) = c'_!\dot{\mathcal{L}}.$$

Using the definitions we see that it is enough to show that the restriction of $\dot{\mathcal{L}}^\sharp$ to $E' - \dot{\mathfrak{g}}_\delta$ is zero. This can be deduced from 1.2(c).

4.4. Let $\mathcal{Q}_\eta^\epsilon(\mathfrak{g}_\delta)$ be the subcategory of $\mathcal{D}(\mathfrak{g}_\delta)$ consisting of complexes which are direct sums of shifts of simple $G_{\underline{0}}$-equivariant perverse sheaves B on $\mathfrak{g}_\delta$ with the following property: there exists a datum $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*)$ as in 4.1(a) and a simple cuspidal perverse sheaf A in $\mathcal{Q}(\mathfrak{l}_\eta)$ such that some shift of B is a direct summand of ${}^\epsilon \mathrm{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)$. We show:

(a) *If* $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*) \in \mathfrak{P}^\epsilon$ *and* A' *is a simple (not necessarily cuspidal) perverse sheaf in* $\mathcal{Q}(\mathfrak{l}_\eta)$, *then* ${}^\epsilon \mathrm{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A') \in \mathcal{Q}_\eta^\epsilon(\mathfrak{g}_\delta)$.

Using [L4, 7.5] we see that some shift of A' is a direct summand of $\mathrm{ind}_{\mathfrak{q}_\eta}^{\mathfrak{l}_\eta}(A)$ for some $\hat{\mathfrak{l}}, \mathfrak{q}$ as in 4.2 where A is a simple cuspidal perverse sheaf in $\mathcal{Q}(\hat{\mathfrak{l}}_\eta)$. It follows that some shift of ${}^\epsilon \mathrm{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A')$ is a direct summand of

(b) $${}^\epsilon \mathrm{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\mathrm{ind}_{\mathfrak{q}_\eta}^{\mathfrak{l}_\eta}(A)).$$

It is then enough to show that the complex (b) belongs to $\mathcal{Q}_\eta^\epsilon(\mathfrak{g}_\delta)$. This follows from the definitions using the transitivity property 4.2(a).

The functor

$${}^\epsilon \mathrm{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta} : \mathcal{Q}(\mathfrak{l}_\eta) \to \mathcal{Q}_\eta^\epsilon(\mathfrak{g}_\delta)$$

(where $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*)$ is as in 4.1(a)) called *spiral induction.*

Let $\mathcal{K}_\eta^\epsilon(\mathfrak{g}_\delta)$ be the abelian group generated by symbols (A), one for each isomorphism class of objects of $\mathcal{Q}_\eta^\epsilon(\mathfrak{g}_\delta)$, subject to the relations $(A) + (A') = (A \oplus A')$ (a Grothendieck group). Now $\mathcal{K}_\eta^\epsilon(\mathfrak{g}_\delta)$ is naturally an $\mathcal{A}$-module by $v^n(A) = (A[n])$ for any $n \in \mathbf{Z}$. We shall write A instead of (A) (in $\mathcal{Q}_\eta^\epsilon(\mathfrak{g}_\delta)$). Clearly, $\mathcal{K}_\eta^\epsilon(\mathfrak{g}_\delta)$ is a free $\mathcal{A}$-module with a finite distinguished basis given by the various simple perverse sheaves in $\mathcal{Q}_\eta^\epsilon(\mathfrak{g}_\delta)$. Now $A, B \mapsto (A : B) = \{A, D(B)\} \in \mathbf{N}((v))$ (see 0.12) defines a pairing

(c) $$(:) : \mathcal{K}_\eta^\epsilon(\mathfrak{g}_\delta) \times \mathcal{K}_\eta^\epsilon(\mathfrak{g}_\delta) \to \mathbf{Z}((v)),$$

which is $\mathcal{A}$-linear in the first argument, $\mathcal{A}$-antilinear in the second argument (for $f \mapsto \bar{f}$) and satisfies $(b_1 : b_2) = \overline{(b_2 : b_1)}$ for all b_1, b_2 in $\mathcal{K}_\eta^\epsilon(\mathfrak{g}_\delta)$.

4.5. In addition to the datum 4.1(a) we consider another datum

(a) $$(\mathfrak{p}'_*, L', P'_0, \mathfrak{l}', \mathfrak{l}'_*, \mathfrak{u}'_*) \in \mathfrak{P}^{\epsilon}$$

such that $\mathfrak{p}_N \subset \mathfrak{p}'_N$ for all $N \in \mathbf{Z}$ and $\mathfrak{p}_N = \mathfrak{p}'_N$ for $N \in \{\eta, -\eta\}$. We then have $\mathfrak{u}'_N \subset \mathfrak{u}_N$ for all $N \in \mathbf{Z}$ and $\mathfrak{u}_N = \mathfrak{u}'_N$ for $N \in \{\eta, -\eta\}$. We also have canonically $\mathfrak{l}_N = \mathfrak{l}'_N$ for $N \in \{\eta, -\eta\}$ and $P_0 \subset P'_0$. Let $\mathcal{P} = P'_0/P_0$. Write $\rho_{\mathcal{P}!}\bar{\mathbf{Q}}_l = \oplus_j \bar{\mathbf{Q}}_l[-2a_j]$ where a_j are integers ≥ 0. (Here $\rho_{\mathcal{P}!}$ is as in 0.12.) Let $A \in \mathcal{Q}(\mathfrak{l}_\eta) = \mathcal{Q}(\mathfrak{l}'_\eta)$. We show:

(b) *Let* $I = {}^\epsilon\operatorname{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)$, $I' = {}^\epsilon\operatorname{Ind}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(A)$. *We have* $I \cong \oplus_j I'[-2a_j]$.

We consider the commutative diagram

$$\begin{array}{ccccccc}
\mathfrak{l}_\eta & \xleftarrow{a} & G_{\underline{0}} \times \mathfrak{p}_\eta & \xrightarrow{b} & E' & \xrightarrow{c} & \mathfrak{g}_\delta \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle 1} \\
\mathfrak{l}'_\eta & \xleftarrow{a'} & G_{\underline{0}} \times \mathfrak{p}'_\eta & \xrightarrow{b'} & \tilde{E}' & \xrightarrow{c'} & \mathfrak{g}_\delta
\end{array},$$

where the upper horizontal maps are as in 4.1(b), the lower horizontal are the analogous maps when 4.1(a) is replaced by (a) and $h : E' \to \tilde{E}'$ is given by $(gP_0, z) \mapsto (gP'_0, z)$. Note that h is a P'_0/P_0-bundle. We can find a complex A_1 (resp. A'_1) on E' (resp. $\tilde{E}'$) such that $I = c_! A_1$, $I' = c'_! A'_1$. We have $A_1 = h^* A'_1$, hence

$$I = c_! A_1 = c'_! h_! A_1 = c'_! h_! h^* A'_1 = c'_!(A'_1 \otimes h_! h^* \bar{\mathbf{Q}}_l) = \oplus_j c'_! A'_1[-2a_j] = \oplus_j I'[-2a_j].$$

This proves (b).

5. Study of a pair of spirals

This section serves as preparation for the next one, which aims to calculate the Hom space between two spiral inductions.

5.1. In addition to $\eta \in \mathbf{Z} - \{0\}$ which has been fixed in 2.9, in this section we fix ϵ', ϵ'' in $\{1, -1\}$. Let

$$(\mathfrak{p}'_*, L', P'_0, \mathfrak{l}', \mathfrak{l}'_*, \mathfrak{u}'_*) \in \mathfrak{P}^{\epsilon'}, \quad (\mathfrak{p}'', L'', P''_0, \mathfrak{l}'', \mathfrak{l}''_*, \mathfrak{u}''_*) \in \mathfrak{P}^{\epsilon''}.$$

We show:

(a) *there exists a splitting* $\tilde{\mathfrak{l}}'_*$ *of* $\mathfrak{p}'_*$ *and a splitting* $\tilde{\mathfrak{l}}''_*$ *of* $\mathfrak{p}''_*$ *such that, if* $\tilde{L}'_0 = e^{\tilde{\mathfrak{l}}'_0} \subset G$ *and* $\tilde{L}''_0 = e^{\tilde{\mathfrak{l}}''_0} \subset G$, *then some maximal torus* $\mathcal{T}$ *of* $G_{\underline{0}}$ *is contained in both* $\tilde{L}'_0$ *and* $\tilde{L}''_0$.

Let $\tilde{\mathfrak{l}}'_*$ be any splitting of $\mathfrak{p}'_*$ and let $\tilde{\mathfrak{l}}''_*$ be any splitting of $\mathfrak{p}''_*$; let $\tilde{L}'_0 = e^{\tilde{\mathfrak{l}}'_0} \subset G$, $\tilde{L}''_0 = e^{\tilde{\mathfrak{l}}''_0} \subset G$. Recall that P'_0 (resp. P''_0) is a parabolic subgroup of $G_{\underline{0}}$ with Levi subgroup $\tilde{L}'_0$ (resp. $\tilde{L}''_0$); hence there exists a maximal torus $\mathcal{T}_0$ of $G_{\underline{0}}$ contained in both P'_0, P''_0. Let ${}'\tilde{L}'_0$ (resp. ${}'\tilde{L}''_0$) be the Levi subgroup of P'_0 (resp. P''_0) such that $\mathcal{T}_0 \subset {}'\tilde{L}'_0$ (resp. $\mathcal{T}_0 \subset {}'\tilde{L}''_0$). We can find $u' \in U_{P'_0}$, $u'' \in U_{P''_0}$ such that $\operatorname{Ad}(u')\tilde{L}'_0 = {}'\tilde{L}'_0$, $\operatorname{Ad}(u'')\tilde{L}''_0 = {}'\tilde{L}''_0$. Now $\{\operatorname{Ad}(u')\tilde{\mathfrak{l}}'_N; N \in \mathbf{Z}\}$ is a splitting of $\{\operatorname{Ad}(u')\mathfrak{p}'_N; N \in \mathbf{Z}\} = \mathfrak{p}'_*$ and $\{\operatorname{Ad}(u'')\tilde{\mathfrak{l}}''_N; N \in \mathbf{Z}\}$ is a splitting of $\{\operatorname{Ad}(u'')\mathfrak{p}''_N; N \in \mathbf{Z}\} = \mathfrak{p}''_*$. Note that $\operatorname{Ad}(u')\tilde{L}'_0$, $\operatorname{Ad}(u'')\tilde{L}''_0$ contain a maximal torus of $G_{\underline{0}}$; (a) is proved.

5.2. Let $(\mathfrak{p}'_*, L', P'_0, \mathfrak{l}', \mathfrak{l}'_*, \mathfrak{u}'_*) \in \mathfrak{P}^{\epsilon'}$, $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*) \in \mathfrak{P}^{\epsilon''}$. Let $A \in \mathcal{Q}(\mathfrak{l}_\eta)$ be a simple cuspidal perverse sheaf. As in 4.3, we have $A = \mathcal{L}^\sharp[\dim \mathfrak{l}_\eta]$ where $\mathcal{L}$ is an irreducible local system on $\overset{\circ}{\mathfrak{l}}_\eta$. Let

$$B = {}^{\epsilon'}\operatorname{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(\mathcal{L}^\sharp).$$

Let $\pi' : \mathfrak{p}'_\eta \to \mathfrak{l}'_\eta$ be the obvious map with kernel $\mathfrak{u}'_\eta$. We want to study the complex $K = \pi'_!(B|_{\mathfrak{p}'_\eta}) \in \mathcal{D}(\mathfrak{l}'_\eta)$. As in 4.3, let

$$\dot{\mathfrak{g}}_\delta = G_{\underline{0}} \overset{P_0}{\times} \pi^{-1}(\overset{\circ}{\mathfrak{l}}_\eta),$$

where $\pi : \mathfrak{p}_\eta \to \mathfrak{l}_\eta$ is the obvious map; let $\dot{\mathcal{L}}$ be the local system on $\dot{\mathfrak{g}}_\delta$ defined in terms of $\mathcal{L}$ as in 4.3. As in 4.3, we define $c' : \dot{\mathfrak{g}}_\delta \to \mathfrak{g}_\delta$ by $c'(g, z) = z$. Let

$$\dot{\mathfrak{p}}'_\eta = \{(gP_0, z) \in G_{\underline{0}}/P_0 \times \mathfrak{p}'_\eta; \operatorname{Ad}(g^{-1})z \in \pi^{-1}(\overset{\circ}{\mathfrak{l}}_\eta)\}.$$

Note that $\dot{\mathfrak{p}}'_\eta$ is the closed subvariety $c'^{-1}\mathfrak{p}'_\eta$ of $\dot{\mathfrak{g}}_\delta$. The restriction of $\dot{\mathcal{L}}$ from $\dot{\mathfrak{g}}_\delta$ to $\dot{\mathfrak{p}}'_\eta$ is denoted again by $\dot{\mathcal{L}}$. Now c' restricts to a map $\dot{\mathfrak{p}}'_\eta \to \mathfrak{p}'_\eta$ whose composition with $\pi' : \mathfrak{p}'_\eta \to \mathfrak{l}'_\eta$ is denoted by $\sigma : \dot{\mathfrak{p}}'_\eta \to \mathfrak{l}'_\eta$. We have $\sigma : (gP_0, z) \mapsto \pi'(z)$. Using 4.3(a) and a proper base change, we see that $K = \sigma_!(\dot{\mathcal{L}})$.

We have a partition $\dot{\mathfrak{p}}'_\eta = \cup_\Omega \dot{\mathfrak{p}}'_{\eta,\Omega}$ into locally closed subvarieties indexed by the various (P'_0, P_0)-double cosets Ω in $G_{\underline{0}}$, where

$$\dot{\mathfrak{p}}'_{\eta,\Omega} = \{(gP_0, z) \in \Omega/P_0 \times \mathfrak{p}'_\eta; \operatorname{Ad}(g^{-1})z \in \pi^{-1}(\overset{\circ}{\mathfrak{l}}_\eta)\}.$$

Let $\sigma_\Omega : \dot{\mathfrak{p}}'_{\eta,\Omega} \to \mathfrak{l}'_\eta$ be the restriction of σ. For any Ω we set

$$K_\Omega = \sigma_{\Omega!}(\dot{\mathcal{L}}|_{\dot{\mathfrak{p}}'_{\eta,\Omega}}) \in \mathcal{D}(\mathfrak{l}'_\eta).$$

We say that Ω is *good* if for some (or equivalently any) $g_0 \in \Omega$, the following condition holds: setting $\mathfrak{p}''_N = \operatorname{Ad}(g_0)\mathfrak{p}_N$, $\mathfrak{u}''_N = \operatorname{Ad}(g_0)\mathfrak{u}_N$ (for $N \in \mathbf{Z}$), the obvious inclusion

$$(\mathfrak{p}'_N \cap \operatorname{Ad}(g_0)\mathfrak{p}_N)/(\mathfrak{p}'_N \cap \operatorname{Ad}(g_0)\mathfrak{u}_N) \to \operatorname{Ad}(g_0)\mathfrak{p}_N/\operatorname{Ad}(g_0)\mathfrak{u}_N$$

is an isomorphism for any $N \in \mathbf{Z}$ that is, $\operatorname{Ad}(g_0)\mathfrak{p}_N = (\mathfrak{p}'_N \cap \operatorname{Ad}(g_0)\mathfrak{p}_N) + \operatorname{Ad}(g_0)\mathfrak{u}_N$. We say that Ω is *bad* if it is not good.

Until the end of 5.4 we fix an Ω as above and we choose $g_0 \in \Omega$. Let $\mathfrak{p}''_N = \operatorname{Ad}(g_0)\mathfrak{p}_N$; then $\mathfrak{p}''_*$ is an ϵ''-spiral. It determines a datum $(\mathfrak{p}''_*, L'', P''_0, \mathfrak{l}'', \mathfrak{l}''_*, \mathfrak{u}''_*) \in \mathfrak{P}^{\epsilon''}$.

By the change of variable $g = hg_0$ we may identify $\dot{\mathfrak{p}}'_{\eta,\Omega}$ with

$$\{(hP''_0, z) \in P'_0P''_0/P''_0 \times \mathfrak{p}'_\eta; \operatorname{Ad}(h^{-1})z \in \operatorname{Ad}(g_0)\pi^{-1}(\overset{\circ}{\mathfrak{l}}_\eta))\}$$

which is the same as

$$\Xi = \{(h(P'_0 \cap P''_0), z) \in P'_0/(P'_0 \cap P''_0) \times \mathfrak{p}'_\eta; \operatorname{Ad}(h^{-1})z \in \pi''^{-1}(\overset{\circ}{\mathfrak{l}}{}''_\eta)\}$$

(in which $\pi'' : \mathfrak{p}''_\eta \to \mathfrak{l}''_\eta$ is the obvious map, with kernel $\mathfrak{u}''_\eta$). In these coordinates, $\sigma_\Omega : \dot{\mathfrak{p}}'_{\eta,\Omega} \to \mathfrak{l}'_\eta$ becomes $(h(P'_0 \cap P''_0), z) \mapsto \pi'(z)$.

We choose a splitting $\tilde{\mathfrak{l}}'_*$ of $\mathfrak{p}'_*$ and a splitting $\tilde{\mathfrak{l}}''_*$ of $\mathfrak{p}''_*$ as in 5.1(a); let $\tilde{L}'_0$, $\tilde{L}''_0$, $\mathcal{T}$ be as in 5.1(a).

Let μ', μ'' be elements of $Y_{G_{\underline{0}},\mathbf{Q}}$ such that $\mathfrak{p}'_* = {}^{\epsilon'}\mathfrak{p}_*^{\mu'}$, $\tilde{\mathfrak{l}}'_* = {}^{\epsilon'}\tilde{\mathfrak{l}}_*^{\mu'}$, $\mathfrak{p}''_* = {}^{\epsilon''}\mathfrak{p}_*^{\mu''}$, $\tilde{\mathfrak{l}}''_* = {}^{\epsilon''}\tilde{\mathfrak{l}}_*^{\mu''}$. Let r', r'' in $\mathbf{Z}_{>0}$ be such that $\lambda' := r'\mu' \in Y_{G_{\underline{0}}}$, $\lambda'' := r''\mu'' \in Y_{G_{\underline{0}}}$.

As in 2.6(d) we have $\lambda'(\mathbf{k}^*) \subset \mathcal{Z}^0_{\tilde{L}'_0}$, $\lambda''(\mathbf{k}^*) \subset \mathcal{Z}^0_{\tilde{L}''_0}$. Since $\mathcal{T}$ is a maximal torus of $\tilde{L}'_0$, we must have $\mathcal{Z}^0_{\tilde{L}'_0} \subset \mathcal{T}$ hence $\lambda'(\mathbf{k}^*) \subset \mathcal{T}$. Similarly, since $\mathcal{T}$ is a maximal torus of $\tilde{L}''_0$, we have $\mathcal{Z}^0_{\tilde{L}''_0} \subset \mathcal{T}$ hence $\lambda''(\mathbf{k}^*) \subset \mathcal{T}$. Since both $\lambda'(\mathbf{k}^*), \lambda''(\mathbf{k}^*)$ are contained in the torus $\mathcal{T}$, we must have $\lambda'(t')\lambda''(t'') = \lambda''(t'')\lambda'(t')$ for any t', t'' in $\mathbf{k}^*$. Hence, if for any k', k'' in $\mathbf{Z}$ and $i \in \mathbf{Z}/m$ we set

$${}_{k',k''}\mathfrak{g}_i = \{x \in \mathfrak{g}_i; \mathrm{Ad}(\lambda'(t'))x = t'^{k'}x, \mathrm{Ad}(\lambda''(t''))x = t''^{k''}x, \quad \forall t', t'' \in \mathbf{k}^*\},$$

then we have $\mathfrak{g} = \oplus_{k',k'',i}({}_{k',k''}\mathfrak{g}_i)$.

For any $N \in \mathbf{Z}$ we have a direct sum decomposition

(a) $$\mathfrak{p}'_N \cap \mathfrak{p}''_N = (\tilde{\mathfrak{l}}'_N \cap \tilde{\mathfrak{l}}''_N) \oplus (\mathfrak{u}'_N \cap \tilde{\mathfrak{l}}''_N) \oplus (\tilde{\mathfrak{l}}'_N \cap \mathfrak{u}''_N) \oplus (\mathfrak{u}'_N \cap \mathfrak{u}''_N).$$

This follows immediately from the decompositions

$$\mathfrak{p}'_N \cap \mathfrak{p}''_N = \oplus_{k',k'';k' \ge Nr'\epsilon', k'' \ge Nr''\epsilon''}({}_{k,k'}\mathfrak{g}_{\underline{N}}),$$
$$\tilde{\mathfrak{l}}'_N \cap \tilde{\mathfrak{l}}''_N = \oplus_{k',k'';k' = Nr'\epsilon', k'' = Nr''\epsilon''}({}_{k,k'}\mathfrak{g}_{\underline{N}}),$$
$$\mathfrak{u}'_N \cap \tilde{\mathfrak{l}}''_N = \oplus_{k',k'';k' > Nr'\epsilon', k'' = Nr''\epsilon''}({}_{k,k'}\mathfrak{g}_{\underline{N}}),$$
$$\tilde{\mathfrak{l}}'_N \cap \mathfrak{u}''_N = \oplus_{k',k'';k' = Nr'\epsilon', k'' > Nr''\epsilon''}({}_{k,k'}\mathfrak{g}_{\underline{N}}),$$
$$\mathfrak{u}'_N \cap \mathfrak{u}''_N = \oplus_{k',k'';k' > Nr'\epsilon', k'' > Nr''\epsilon''}({}_{k,k'}\mathfrak{g}_{\underline{N}}).$$

For $N \in \mathbf{Z}$ let $\mathfrak{q}''_N$ be the image of $\mathfrak{p}'_N \cap \mathfrak{p}''_N$ under the obvious map $\mathfrak{p}''_N \to \mathfrak{l}''_N$; let $\mathfrak{q}'' = \oplus_{N \in \mathbf{Z}}(\mathfrak{q}''_N)$, a Lie subalgebra of $\mathfrak{l}''$. We show:

(b) *$\mathfrak{q}''$ is a parabolic subalgebra of $\mathfrak{l}''$ compatible with the $\mathbf{Z}$-grading of $\mathfrak{l}''$.*

For $N \in \mathbf{Z}$ we set $\tilde{\mathfrak{q}}''_N = \tilde{\mathfrak{l}}''_N \cap \mathfrak{p}'_N$. Let $\tilde{\mathfrak{q}}'' = \oplus_{N \in \mathbf{Z}}\tilde{\mathfrak{q}}''_N$, a Lie subalgebra of $\tilde{\mathfrak{l}}''$. From (a) we see that the obvious isomorphism $\tilde{\mathfrak{l}}'' \xrightarrow{\sim} \mathfrak{l}''$ carries $\tilde{\mathfrak{q}}''$ to $\mathfrak{q}''$. Hence (b) follows from (c) below:

(c) *$\tilde{\mathfrak{q}}''$ is a parabolic subalgebra of $\tilde{\mathfrak{l}}''$ compatible with the $\mathbf{Z}$-grading of $\tilde{\mathfrak{l}}''$.*

We have

$$\tilde{\mathfrak{q}}'' = \oplus_{k', N \in \mathbf{Z}; k' \ge Nr'\epsilon'}({}_{k', Nr''\epsilon''}\mathfrak{g}_{\underline{N}}).$$

We define $\lambda_1 \in Y_{\tilde{L}''}$ by $\lambda_1(t) = \lambda'(t^{r''})\lambda''(t^{-r'\epsilon'\epsilon''})$ for all $t \in \mathbf{k}^*$. Then $\mathrm{Ad}(\lambda_1(t))$ acts on the subspace ${}_{k',Nr''\epsilon''}\mathfrak{g}_{\underline{N}}$ of $\tilde{\mathfrak{l}}''$ as $t^{k'r''-r'r''N\epsilon'}$; the last exponent of t is ≥ 0 if and only if $k' \ge r'N\epsilon'$ which is just the condition that ${}_{k',Nr''\epsilon''}\mathfrak{g}_{\underline{N}}$ is one of the summands in the direct sum decomposition of $\tilde{\mathfrak{q}}''$. This proves (c).

For $N \in \mathbf{Z}$ let $\mathfrak{q}'_N$ be the image of $\mathfrak{p}'_N \cap \mathfrak{p}''_N$ under the obvious map $\mathfrak{p}'_N \to \mathfrak{l}'_N$; let $\mathfrak{q}' = \oplus_{N \in \mathbf{Z}}\mathfrak{q}'_N$, a Lie subalgebra of $\mathfrak{l}'$.

For $N \in \mathbf{Z}$ we set $\tilde{\mathfrak{q}}'_N = \tilde{\mathfrak{l}}'_N \cap \mathfrak{p}''_N$. Let $\tilde{\mathfrak{q}}' = \oplus_{N \in \mathbf{Z}}\tilde{\mathfrak{q}}'_N$, a Lie subalgebra of $\tilde{\mathfrak{l}}'$. The following result is proved in the same way as (b),(c).

(d) *$\mathfrak{q}'$ is a parabolic subalgebra of $\mathfrak{l}'$ compatible with the $\mathbf{Z}$-grading of $\mathfrak{l}'$; $\tilde{\mathfrak{q}}'$ is a parabolic subalgebra of $\tilde{\mathfrak{l}}'$ compatible with the $\mathbf{Z}$-grading of $\tilde{\mathfrak{l}}'$.*

We set ${}^!\tilde{\mathfrak{q}}'' = \oplus_N {}^!\tilde{\mathfrak{q}}''_N$, ${}^!\tilde{\mathfrak{q}}' = \oplus_N({}^!\tilde{\mathfrak{q}}'_N)$, where

$${}^!\tilde{\mathfrak{q}}''_N = \oplus_{k' \in \mathbf{Z}; k' > Nr'\epsilon'}({}_{k',Nr''\epsilon''}\mathfrak{g}_{\underline{N}}), \quad {}^!\tilde{\mathfrak{q}}'_N = \oplus_{k'' \in \mathbf{Z}; k'' > Nr''\epsilon''}({}_{Nr'\epsilon',k''}\mathfrak{g}_{\underline{N}}).$$

The proof of (c) shows also that ${}^!\tilde{\mathfrak{q}}''$ is the nilradical of $\tilde{\mathfrak{q}}''$ and that

$$\oplus_{N \in \mathbf{Z}}({}_{Nr'\epsilon',Nr''\epsilon''}\mathfrak{g}_{\underline{N}})$$

is a Levi subalgebra of $\tilde{\mathfrak{q}}''$. Similarly, ${}^!\tilde{\mathfrak{q}}'$ is the nilradical of $\tilde{\mathfrak{q}}'$ and

$$\oplus_{N \in \mathbf{Z}}({}_{Nr'\epsilon',Nr''\epsilon''}\mathfrak{g}_{\underline{N}})$$

is a Levi subalgebra of $\tilde{\mathfrak{q}}''$. Thus,

(e) *$\tilde{\mathfrak{q}}', \tilde{\mathfrak{q}}''$ have a common Levi subalgebra, namely* $\oplus_{N\in\mathbf{Z}}({}_{Nr'\epsilon',Nr''\epsilon''}\mathfrak{g}_{\underline{N}})$.

5.3. In this subsection we assume that Ω is bad. Then for some N, $\tilde{\mathfrak{l}}''_N \cap \mathfrak{p}'_N$ is strictly contained in $\tilde{\mathfrak{l}}''_N$. Hence $\tilde{\mathfrak{q}}''$ is a proper parabolic subalgebra of $\tilde{\mathfrak{l}}''$ (see 5.2(c)). We will show that

$$K_\Omega = \sigma_{\Omega !}(\dot{\mathcal{L}}|_{\dot{\mathfrak{p}}'_{\eta,\Omega}}) = 0 \in \mathcal{D}(\mathfrak{l}'_\eta). \tag{a}$$

This is equivalent to the following statement:

(b) *for any $y \in \tilde{\mathfrak{l}}'_\eta$, the cohomology groups H^j_c of the variety*

$$\{(h(P'_0 \cap P''_0), z) \in P'_0/(P'_0 \cap P''_0) \times \mathfrak{p}'_\eta; z - y \in \mathfrak{u}'_\eta, \mathrm{Ad}(h^{-1})z \in \pi''^{-1}(\overset{\circ}{\mathfrak{l}''_\eta})\}$$

with coefficients in the local system defined by $\dot{\mathcal{L}}$, are zero for all $j \in \mathbf{Z}$.

(We have identified $\tilde{\mathfrak{l}}'_\eta$, $\mathfrak{l}'_\eta$ via π'.) Considering the fibers of the first projection of the last variety to $P'_0/(P'_0 \cap P''_0)$, we see that it suffices to show that:

(c) *for any $h \in P'_0$ and any $y \in \tilde{\mathfrak{l}}'_\eta$, the cohomology groups H^j_c of the variety*

$$\{z \in \mathfrak{p}'_\eta; z - y \in \mathfrak{u}'_\eta, \mathrm{Ad}(h^{-1})z \in \overset{\circ}{\tilde{\mathfrak{l}}''_\eta} + \mathfrak{u}''_\eta\}$$

with coefficients in the local system defined by $\dot{\mathcal{L}}$, are zero for all $j \in \mathbf{Z}$.

(We have used that $\pi''^{-1}(\overset{\circ}{\mathfrak{l}''_\eta}) = \overset{\circ}{\tilde{\mathfrak{l}}''_\eta} + \mathfrak{u}''_\eta$.)

If z is as in (c), then we have automatically $\mathrm{Ad}(h^{-1})z \in \mathfrak{p}'_\eta$; since $\overset{\circ}{\tilde{\mathfrak{l}}''_\eta} + \mathfrak{u}''_\eta \subset \mathfrak{p}''_\eta$, the condition that $\mathrm{Ad}(h^{-1})z \in \overset{\circ}{\tilde{\mathfrak{l}}''_\eta} + \mathfrak{u}''_\eta$ implies $\mathrm{Ad}(h^{-1})z \in \mathfrak{p}'_\eta \cap \mathfrak{p}''_\eta$. By 5.2(a), we can then write uniquely $\mathrm{Ad}(h^{-1})z = \gamma + \nu' + \nu'' + \mu$, where

$$\gamma \in \tilde{\mathfrak{l}}'_\eta \cap \tilde{\mathfrak{l}}''_\eta, \nu' \in \mathfrak{u}'_\eta \cap \tilde{\mathfrak{l}}''_\eta, \nu'' \in \tilde{\mathfrak{l}}'_\eta \cap \mathfrak{u}''_\eta, \mu \in \mathfrak{u}'_\eta \cap \mathfrak{u}''_\eta. \tag{e}$$

The condition that $\mathrm{Ad}(h^{-1})z \in \overset{\circ}{\tilde{\mathfrak{l}}''_\eta} + \mathfrak{u}''_\eta$ can be expressed as $\gamma + \nu' \in \overset{\circ}{\tilde{\mathfrak{l}}''_\eta}$. The condition that $z - y \in \mathfrak{u}'_\eta$ is equivalent to $\mathrm{Ad}(h^{-1})z - \mathrm{Ad}(h^{-1})y \in \mathfrak{u}'_\eta$ or (if we define $y' \in \tilde{\mathfrak{l}}'_\eta$ by $\mathrm{Ad}(h^{-1})y - y' \in \mathfrak{u}'_\eta$) to $\gamma + \nu'' = y'$. Note that y', γ, ν'' are uniquely determined by h, y. Hence the variety in (c) can be identified with

$$(\gamma + (\mathfrak{u}'_\eta \cap \tilde{\mathfrak{l}}''_\eta)) \cap \overset{\circ}{\tilde{\mathfrak{l}}''_\eta}) \times (\mathfrak{u}'_\eta \cap \mathfrak{u}''_\eta).$$

Under this identification, the local system $\dot{\mathcal{L}}$ is the pullback of $\mathcal{L}$ (viewed as a local system on $\overset{\circ}{\tilde{\mathfrak{l}}''_\eta}$) from the first factor. Now the desired vanishing of cohomology follows from the vanishing property [L4, 4.4(c)] of $\mathcal{L}$, since in our case $\tilde{\mathfrak{q}}'' = \oplus_N(\tilde{\mathfrak{l}}''_N \cap \mathfrak{p}'_N)$ is a proper parabolic subalgebra of $\tilde{\mathfrak{l}}''$ with nilradical $\oplus_N(\tilde{\mathfrak{l}}''_N \cap \mathfrak{u}'_N)$.

5.4. In this subsection we assume that Ω is good. Then for any N we have $\tilde{\mathfrak{l}}''_N \cap \mathfrak{p}'_N = \tilde{\mathfrak{l}}''_N$ that is, $\tilde{\mathfrak{l}}''_N \subset \mathfrak{p}'_N$. We also have $\tilde{\mathfrak{q}}'' = \tilde{\mathfrak{l}}''$. Thus $\tilde{\mathfrak{q}}''$ is reductive so it is equal to its Levi subalgebra $\oplus_{N\in\mathbf{Z}}({}_{Nr'\epsilon',Nr''\epsilon''}\mathfrak{g}_{\underline{N}})$ (see 5.2(e)) which is then equal to $\tilde{\mathfrak{l}}''$ and is also a Levi subalgebra of $\tilde{\mathfrak{q}}'$ (see 5.2(e)). Thus,

(a) *$\tilde{\mathfrak{l}}''$ is a Levi subalgebra of $\tilde{\mathfrak{q}}'$.*

Now $\mathrm{Ad}(g_0)$ defines an isomorphism $\mathfrak{l} \xrightarrow{\sim} \mathfrak{l}''$. Composing this with the inverse of the obvious isomorphism $\tilde{\mathfrak{l}}'' \xrightarrow{\sim} \mathfrak{l}''$ we obtain an isomorphism of $\mathbf{Z}$-graded Lie algebras $\mathfrak{l} \xrightarrow{\sim} \tilde{\mathfrak{l}}''$. Using this, we can transport $\mathcal{L}$ (a local system on $\overset{\circ}{\mathfrak{l}}_\eta$; see 5.1) to a local system $\mathcal{L}''$ on $\overset{\circ}{\tilde{\mathfrak{l}}}{}''_\eta$. Let $\mathcal{L}''^\sharp \in \mathcal{D}(\tilde{\mathfrak{l}}''_\eta)$ be as in 0.11. Then

$$\operatorname{ind}_{\tilde{\mathfrak{q}}'_\eta}^{\tilde{\mathfrak{l}}'_\eta}(\mathcal{L}''^\sharp) \in \mathcal{Q}(\tilde{\mathfrak{l}}'_\eta)$$

is defined as in 1.3 (we identify $\tilde{\mathfrak{l}}''$ with the reductive quotient of $\tilde{\mathfrak{q}}'$; see (a)). We now state the following result.

(b) *We have* $K_\Omega = \operatorname{ind}_{\tilde{\mathfrak{q}}'_\eta}^{\tilde{\mathfrak{l}}'_\eta}(\mathcal{L}''^\sharp)[-2f]$, *where*

$$f = \dim \mathfrak{u}'_0 - \dim(\mathfrak{u}'_0 \cap \mathfrak{p}''_0) + \dim(\mathfrak{u}'_\eta \cap \mathfrak{u}''_\eta).$$

Let $\tilde{Q}'_0 = e^{\tilde{\mathfrak{q}}'_0} \subset \tilde{L}'_0$, a parabolic subgroup of $\tilde{L}'_0$. Let

$$\Xi' = \{('h\tilde{Q}'_0, 'z) \in \tilde{L}'_0/\tilde{Q}'_0 \times \tilde{\mathfrak{l}}'_\eta; \operatorname{Ad}('h^{-1})'z \in \overset{\circ}{\tilde{\mathfrak{l}}}{}''_\eta + {}^!\tilde{\mathfrak{q}}_\eta\}.$$

Define $c'' : \Xi' \to \tilde{\mathfrak{l}}'_\eta$ by $c''('hQ'_0, 'z) = 'z$. By the argument in [L4, 6.6] (for $\tilde{L}'$ instead of G) we have

(c) $$\operatorname{ind}_{\tilde{\mathfrak{q}}'_\eta}^{\tilde{\mathfrak{l}}'_\eta}(\mathcal{L}''^\sharp) = c''_! \dot{\mathcal{L}}'',$$

where $\dot{\mathcal{L}}''$ is a certain local system on Ξ' determined by $\mathcal{L}''$ and such that $\Delta^*\dot{\mathcal{L}}'' = \dot{\mathcal{L}}$ where $\Delta : \Xi \to \Xi'$ (Ξ as in 5.2) is the map induced by the canonical maps $P'_0 \to \tilde{L}'_0$ (with kernel $U_{P'_0}$) and $\mathfrak{p}'_\eta \to \tilde{\mathfrak{l}}'_\eta$ (with kernel $\mathfrak{u}'_\eta$); $\dot{\mathcal{L}}$ is the local system on Ξ considered in 5.2. We consider the following statement:

(d) Δ *is an affine space bundle with fibers of dimension* f.

Assuming that (d) holds, we have

$$K_\Omega = c''_! \Delta_! \dot{\mathcal{L}} = c''_! \dot{\mathcal{L}}'' \otimes \Delta_! \bar{\mathbf{Q}}_l = c''_! \dot{\mathcal{L}}''[-2f]$$

and we see that (b) follows from (c). It remains to prove (d).

Let $'h \in \tilde{L}'_0$, $'z \in \tilde{\mathfrak{l}}'_\eta$ be such that $('hQ'_0, 'z) \in \Xi'$. Setting $h = 'hu$, $z = 'z + \tilde{z}$, we see that $\Delta^{-1}('hQ'_0, 'z)$ can be identified with

$$\{(u(U_{P'_0} \cap P''_0), \tilde{z}) \in (U_{P'_0}/(U_{P'_0} \cap P''_0)) \times \mathfrak{u}'_\eta; \operatorname{Ad}(u^{-1})\operatorname{Ad}('h^{-1})('z + \tilde{z}) \in \overset{\circ}{\tilde{\mathfrak{l}}}{}''_\eta + \mathfrak{u}''_\eta\}.$$

It suffices to show that

(e) $$\{(u, \tilde{z}) \in U_{P'_0} \times \mathfrak{u}'_\eta; \operatorname{Ad}(u^{-1})\operatorname{Ad}('h^{-1})('z + \tilde{z}) \in \overset{\circ}{\tilde{\mathfrak{l}}}{}''_\eta + \mathfrak{u}''_\eta\}$$

is isomorphic to $U_{P'_0} \times (\mathfrak{u}'_\eta \cap \mathfrak{u}''_\eta)$. If $(u, \tilde{z})$ are as in (e), we have automatically $\operatorname{Ad}(u^{-1})\operatorname{Ad}('h^{-1})('z + \tilde{z}) \in \mathfrak{p}'_\eta$ (since $'z + \tilde{z} \in \mathfrak{p}'_\eta$ and $'hu \in P'_0$). Setting $\operatorname{Ad}('h^{-1})'z = a \in \overset{\circ}{\tilde{\mathfrak{l}}}{}''_\eta + {}^!\tilde{\mathfrak{q}}_\eta$ (where a is fixed) and $\operatorname{Ad}(u^{-1})\operatorname{Ad}('h^{-1})\tilde{z} = \tilde{z}' \in \mathfrak{u}'_\eta$, we see that the variety (e) may be identified with the variety

(f) $$\{(u, \tilde{z}') \in U_{P'_0} \times \mathfrak{u}'_\eta; \operatorname{Ad}(u^{-1})a + \tilde{z}' \in \overset{\circ}{\tilde{\mathfrak{l}}}{}''_\eta + (\mathfrak{p}'_\eta \cap \mathfrak{u}''_\eta)\}.$$

By 5.2(a) we can write uniquely

$$\operatorname{Ad}(u^{-1})a + \tilde{z}' = \gamma + \nu + \mu,$$

where $\gamma \in \overset{\circ}{\tilde{\mathfrak{l}}}{}''_\eta$, $\nu \in \tilde{\mathfrak{l}}'_\eta \cap \mathfrak{u}''_\eta$, $\mu \in \mathfrak{u}'_\eta \cap \mathfrak{u}''_\eta$. Setting $\hat{z} = \mu - \tilde{z}$ we see that (f) can be identified with the variety of all quintuples $(u, \hat{z}, \gamma, \nu, \nu')$ in

$$U_{P'_0} \times \mathfrak{u}'_\eta \times \overset{\circ}{\tilde{\mathfrak{l}}}{}''_\eta \times (\tilde{\mathfrak{l}}'_\eta \cap \mathfrak{u}''_\eta) \times (\mathfrak{u}'_\eta \cap \mathfrak{u}''_\eta)$$

such that

$$\text{(g)} \qquad \mathrm{Ad}(u^{-1})a = \gamma + \nu + \hat{z}.$$

Since $a \in \tilde{\mathfrak{l}}'_\eta$, we have $\mathrm{Ad}(u^{-1})a - a \in \mathfrak{u}'_\eta$ for $u \in U_{P'_0}$. Hence in (g) we have $\gamma + \nu = a$ and $\hat{z} = \mathrm{Ad}(u^{-1})a - a$. In particular, γ, ν are uniquely determined. Thus, our variety may be identified with $U_{P'_0} \times (\mathfrak{u}'_N \cap \mathfrak{u}''_N)$. This completes the proof of (d), hence that of (b).

5.5. From the results in 5.3 and 5.4 we can deduce, using the argument in [L4, 8.9] (based on [L4, 1.4]), the following result.

Proposition 5.6. *We have $K \in \mathcal{Q}(\mathfrak{l}'_\eta)$; moreover, we have (noncanonically) $K \cong \oplus_\Omega K_\Omega$, where Ω runs over good (P'_0, P_0)-double cosets in $G_{\underline{0}}$.*

6. Spiral restriction

We introduce the spiral restriction functor which is adjoint to the spiral induction. The main result in this section is Proposition 6.4, which calculates the inner product $\{,\}$ (in the sense of 0.12) of two spiral inductions.

6.1. **Definition of spiral restriction.** In addition to $\eta \in \mathbf{Z} - \{0\}$ which has been fixed in 2.9, in this section we fix ϵ', ϵ'' in $\{1, -1\}$. Let $(\mathfrak{p}'_*, L', P'_0, \mathfrak{l}', \mathfrak{l}'_*, \mathfrak{u}'_*) \in \mathfrak{P}^{\epsilon'}$. Let $\pi' : \mathfrak{p}'_\eta \to \mathfrak{l}'_\eta$ be the obvious map. For any $B \in \mathcal{D}(\mathfrak{g}_\delta)$ we set

$${}^{\epsilon'}\mathrm{Res}^{\mathfrak{g}_\delta}_{\mathfrak{p}'_\eta}(B) = \pi'_!(B|_{\mathfrak{p}'_\eta}) \in \mathcal{D}(\mathfrak{l}'_\eta).$$

We show:

(a) *If $B \in \mathcal{Q}^{\epsilon''}_\eta(\mathfrak{g}_\delta)$, then ${}^{\epsilon'}\mathrm{Res}^{\mathfrak{g}_\delta}_{\mathfrak{p}'_\eta}(B) \in \mathcal{Q}(\mathfrak{l}'_\eta)$.*

To prove this we can assume that B is in addition a simple perverse sheaf. Then, using the definition of $\mathcal{Q}^{\epsilon''}_\eta(\mathfrak{g}_\delta)$, we see that it is enough to prove (a) in the case where $B = {}^{\epsilon''}\mathrm{Ind}^{\mathfrak{g}_\delta}_{\mathfrak{p}_\eta}(\mathcal{L}^\sharp)$, with $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*) \in \mathfrak{P}^{\epsilon''}$, $\mathcal{L}^\sharp$ as in 5.2. In this case, (a) follows from 5.6.

We thus have a functor ${}^{\epsilon'}\mathrm{Res}^{\mathfrak{g}_\delta}_{\mathfrak{p}'_\eta} : \mathcal{Q}^{\epsilon''}_\eta(\mathfrak{g}_\delta) \to \mathcal{Q}(\mathfrak{l}'_\eta)$ called *spiral restriction.*

We have the following result.

Proposition 6.2 ((Adjunction))**.** *Let $C \in \mathcal{Q}(\mathfrak{l}'_\eta)$, and let $B \in \mathcal{Q}^{\epsilon''}_\eta(\mathfrak{g}_\delta)$. For any $j \in \mathbf{Z}$ we have*

$$\text{(a)} \qquad d_j(\mathfrak{l}'_\eta; C, {}^{\epsilon'}\mathrm{Res}^{\mathfrak{g}_\delta}_{\mathfrak{p}'_\eta}(B)) = d_{j'}(\mathfrak{g}_\delta; {}^{\epsilon'}\mathrm{Ind}^{\mathfrak{g}_\delta}_{\mathfrak{p}'_\eta}(C), B),$$

where $j' = j + 2\dim \mathfrak{u}'_0$.

The proof is almost a copy of that of [L4, 9.2]. We omit it.

For $B \in \mathcal{D}(\mathfrak{g}_\delta)$ we set

$$ {}^{\epsilon'}\widetilde{\mathrm{Res}}^{\mathfrak{g}_\delta}_{\mathfrak{p}'_\eta}(B) = {}^{\epsilon'}\mathrm{Res}^{\mathfrak{g}_\delta}_{\mathfrak{p}'_\eta}(B)[\dim \mathfrak{u}'_\eta - \dim \mathfrak{u}'_0]. $$

With this notation, the equality (a) can be reformulated without a shift from j to j' as follows:

(b) $$ d_j(\mathfrak{l}'_\eta; C, {}^{\epsilon'}\widetilde{\mathrm{Res}}^{\mathfrak{g}_\delta}_{\mathfrak{p}'_\eta}(B)) = d_j(\mathfrak{g}_\delta; {}^{\epsilon'}\widetilde{\mathrm{Ind}}^{\mathfrak{g}_\delta}_{\mathfrak{p}'_\eta}(C), B). $$

6.3. Let $(\mathfrak{p}'_*, L', P'_0, \mathfrak{l}', \mathfrak{l}'_*, \mathfrak{u}'_*) \in \mathfrak{P}^{\epsilon'}$, $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*) \in \mathfrak{P}^{\epsilon''}$. Let $A \in \mathcal{Q}(\mathfrak{l}_\eta)$, $A' \in \mathcal{Q}(\mathfrak{l}'_\eta)$ be cuspidal perverse sheaves. As in 4.3 we have $A = \mathcal{L}^\sharp[\dim \mathfrak{l}_\eta]$, $A' = \mathcal{L}'^\sharp[\dim \mathfrak{l}'_\eta]$ where $\mathcal{L}$ (resp. $\mathcal{L}'$) is a local system on $\overset{\circ}{\mathfrak{l}}_\eta$ (resp. $\overset{\circ}{\mathfrak{l}}{}'_\eta$).

We denote by X the set of all $g \in G_{\underline{0}}$ such that the ϵ''-spiral $\{\mathrm{Ad}(g)\mathfrak{p}_N; N \in \mathbf{Z}\}$ and the ϵ'-spiral $\mathfrak{p}'_*$ have a common splitting. If $g \in X$ there is a unique isomorphism of $\mathbf{Z}$-graded Lie algebras $\lambda_g : \mathfrak{l} \to \mathfrak{l}'$ such that the compositions

$$ \mathrm{Ad}(g)\mathfrak{p}_N \cap \mathfrak{p}'_N \to \mathfrak{p}'_N \to \mathfrak{l}'_N, $$

$$ \mathrm{Ad}(g)\mathfrak{p}_N \cap \mathfrak{p}'_N \xrightarrow{\mathrm{Ad}(g^{-1})} \mathfrak{p}_N \to \mathfrak{l}_N \xrightarrow{\lambda_g} \mathfrak{l}'_N $$

coincide for any N (the unnamed maps are the obvious imbeddings or projections). Moreover, λ_g is induced by an isomorphism $L \to L'$. Let X' be the set of all $g \in X$ such that $\lambda_g : \mathfrak{l}_\eta \xrightarrow{\sim} \mathfrak{l}'_\eta$ carries $\mathcal{L}$ to the dual of $\mathcal{L}'$. For any $g \in X'$ we set

$$ \tau(g) = -\dim \frac{\mathfrak{u}'_0 + \mathrm{Ad}(g)\mathfrak{u}_0}{\mathfrak{u}'_0 \cap \mathrm{Ad}(g)\mathfrak{u}_0} + \dim \frac{\mathfrak{u}'_\eta + \mathrm{Ad}(g)\mathfrak{u}_\eta}{\mathfrak{u}'_\eta \cap \mathrm{Ad}(g)\mathfrak{u}_\eta}. $$

Note that both X and X' are unions of (P'_0, P_0)-double cosets in $G_{\underline{0}}$ and that $\tau(g)$ depends only on the double coset of g. We have the following result.

Proposition 6.4. *Let*

$$ \Pi = \sum_{j \in \mathbf{Z}} d_j(\mathfrak{g}_\delta; {}^{\epsilon'}\widetilde{\mathrm{Ind}}^{\mathfrak{g}_\delta}_{\mathfrak{p}'_\eta}(A'), {}^{\epsilon''}\widetilde{\mathrm{Ind}}^{\mathfrak{g}_\delta}_{\mathfrak{p}_\eta}(A))v^{-j} \in \mathbf{N}((v)). $$

We have

$$ \Pi = (1 - v^2)^{-r} \sum_{g_0} v^{\tau(g_0)}, $$

where r is the dimension of the center of $\mathfrak{l}$ and the sum is taken over a set of representatives g_0 for the (P'_0, P_0)-double cosets in $G_{\underline{0}}$ that are contained in X'. In particular, if $\Pi \neq 0$, then $X' \neq \emptyset$.

Using 6.2, we have

$$ \Pi = \sum_{j \in \mathbf{Z}} d_j(\mathfrak{l}'_\eta; A', {}^{\epsilon'}\widetilde{\mathrm{Res}}^{\mathfrak{g}_\delta}_{\mathfrak{p}'_\eta}({}^{\epsilon''}\widetilde{\mathrm{Ind}}^{\mathfrak{g}_\delta}_{\mathfrak{p}_\eta}(A)))v^{-j} = \sum_{j \in \mathbf{Z}} d_{j+s}(\mathfrak{l}'_\eta; A', K)v^{-j}, $$

where $s = \dim \mathfrak{u}_0 + \dim \mathfrak{u}_\eta + \dim \mathfrak{u}'_\eta - \dim \mathfrak{u}'_0 + \dim \mathfrak{l}_\eta$ and

$$ K = {}^{\epsilon'}\mathrm{Res}^{\mathfrak{g}_\delta}_{\mathfrak{p}'_\eta}({}^{\epsilon''}\mathrm{Ind}^{\mathfrak{g}_\delta}_{\mathfrak{p}_\eta}(\mathcal{L}^\sharp)) $$

is as in 5.2. Using the description of K in 5.3(a), 5.4(b), 5.6, we see that

(a) $$ \Pi = \sum_{j \in \mathbf{Z}} \sum_{g} Q_j(g) v^{-j+s-2f(g)}, $$

where g runs over a set of representatives for the (P'_0, P_0)-double cosets in $G_{\underline{0}}$ which are good (see 5.2) and

$$Q_j(g) = d_j(\tilde{\mathfrak{l}}'_\eta; A', \mathrm{ind}_{\tilde{\mathfrak{q}}'_\eta}^{\tilde{\mathfrak{l}}'_\eta}(\mathcal{L}''^\sharp)),$$

$$f(g) = \dim(\mathfrak{u}'_0/(\mathfrak{u}'_0 \cap \mathrm{Ad}(g)\mathfrak{p}_0) + \dim(\mathfrak{u}'_\eta \cap \mathrm{Ad}(g)\mathfrak{u}_\eta);$$

the following notation is used:

$\tilde{\mathfrak{l}}'_*$ is a certain splitting of $\mathfrak{p}'_*$, $\tilde{\mathfrak{l}}''_*$ is a certain splitting of $\{\mathrm{Ad}(g)\mathfrak{p}_N; N \in \mathbf{Z}\}$, $\tilde{\mathfrak{q}}' = \oplus_{N\in\mathbf{Z}}\tilde{\mathfrak{q}}'_N$ (where $\tilde{\mathfrak{q}}'_N = \tilde{\mathfrak{l}}'_N \cap \mathrm{Ad}(g_0)\mathfrak{p}_N$) is a parabolic subalgebra of $\tilde{\mathfrak{l}}' = \oplus_N \tilde{\mathfrak{l}}'_N$ whose with Levi subalgebra $\tilde{\mathfrak{l}}'' = \oplus_N \tilde{\mathfrak{l}}''_N$; A' is viewed as an object of $\mathcal{Q}(\tilde{\mathfrak{l}}'_\eta)$ via the obvious isomorphism $\tilde{\mathfrak{l}}'_\eta \to \mathfrak{l}'_\eta$ and $\mathcal{L}''^\sharp \in \mathcal{Q}(\tilde{\mathfrak{l}}''_\eta)$ corresponds to $\mathcal{L}^\sharp$ via the isomorphism $\mathfrak{l}_\eta \xrightarrow{\mathrm{Ad}(g)} \mathrm{Ad}(g)\mathfrak{p}_\eta/\mathrm{Ad}(g)\mathfrak{u}_\eta = \tilde{\mathfrak{l}}''_\eta$.

By the implication (a) $\Longrightarrow$ (c) in [L4, 10.6], we have $Q_j(g) = 0$ unless $\tilde{\mathfrak{q}}' = \tilde{\mathfrak{l}}'$. In this case, since $\tilde{\mathfrak{l}}''$ is a Levi subalgebra of $\tilde{\mathfrak{q}}'$, we must have $\tilde{\mathfrak{l}}' = \tilde{\mathfrak{l}}''$ so that $g \in X$. Conversely, if $g \in X$, then the (P'_0, P_0)-double coset of g is good. Indeed, let $\tilde{\mathfrak{l}}'_*$ be a splitting of $\mathfrak{p}'_*$ which is also a splitting for $\{\mathrm{Ad}(g)\mathfrak{p}_N; N \in \mathbf{Z}\}$. We have

$$\mathrm{Ad}(g)\mathfrak{p}_N = \tilde{\mathfrak{l}}'_N \oplus \mathrm{Ad}(g)\mathfrak{u}_N \subset (\mathfrak{p}'_N \cap \mathrm{Ad}(g)\mathfrak{p}_N) + \mathrm{Ad}(g)\mathfrak{u}_N \subset \mathrm{Ad}(g)\mathfrak{p}_N$$

and our claim follows. Thus the sum in (a) can be taken over a set of representatives g for the (P'_0, P_0)-double cosets in $G_{\underline{0}}$ that are contained in X and for such g we have $Q_j(g) = d_j(\tilde{\mathfrak{l}}'_\eta; A', \mathcal{L}''^\sharp)$ where $\tilde{\mathfrak{l}}' = \tilde{\mathfrak{l}}''$, $\mathcal{L}''^\sharp \in \mathcal{Q}(\tilde{\mathfrak{l}}''_\eta)$ are as above. Using [L4, 15.1], we see that in the sum over g in (a) we can take $g \in X'$ and that the contribution of such g to the sum is $(1 - v^2)^{-r} v^{s-2f(g)-d}$ where $d = \dim \mathfrak{l}_\eta$. It remains to show that for g as above we have $s - 2f(g) - d = \tau(g)$. It is enough to show that:

(b) $\mathfrak{u}'_0 \cap \mathrm{Ad}(g)\mathfrak{p}_0 = \mathfrak{u}'_0 \cap \mathrm{Ad}(g)\mathfrak{u}_0$,

(c) $\dim(\mathrm{Ad}(g)\mathfrak{u}_0) = \dim \mathfrak{u}'_0$.

Now (b),(c) hold since $\mathrm{Ad}(g)\mathfrak{p}_0$, $\mathfrak{p}'_0$ are parabolic subalgebras of $\mathfrak{g}_{\underline{0}}$ with nilradicals $\mathrm{Ad}(g)\mathfrak{u}_0$, $\mathfrak{u}'_0$ and with a common Levi subalgebra. This completes the proof of the proposition.

6.5. In the special case where

$$(\mathfrak{p}'_*, L', P'_0, \mathfrak{l}', \mathfrak{l}'_*, \mathfrak{u}'_*) = (\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*)$$

and $A' \cong D(A)$, the sum $\sum_g v^{\tau(g)}$ in Proposition 6.4 is over a nonempty set of g (we have $1 \in X'$) hence the sum is nonzero and Π in 6.4 is nonzero. In particular, we see that

(a) $$^{\epsilon''}\widetilde{\mathrm{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A) \neq 0.$$

6.6. **The map ψ from simple perverse sheaves to $\underline{\mathfrak{T}}_\eta$.** Let B be a simple perverse sheaf in $\mathcal{Q}_\eta^{\epsilon''}(\mathfrak{g}_\delta)$. We associate to B an element of $\underline{\mathfrak{T}}_\eta$ (see 3.5) as follows. We can find, $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*) \in \mathfrak{P}^{\epsilon''}$ and A as in 6.3 such that

$$^{\epsilon''}\widetilde{\mathrm{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A) \cong B[d] \oplus C,$$

where $d \in \mathbf{Z}$ and $C \in \mathcal{Q}_\eta^{\epsilon''}(\mathfrak{g}_\delta)$. Let $\tilde{\mathfrak{l}}_*$ be a splitting of $\mathfrak{p}_*$. Let $\tilde{\mathfrak{l}} = \oplus_N \tilde{\mathfrak{l}}_N$, $\tilde{L} = e^{\tilde{\mathfrak{l}}} \subset G$, $\tilde{L}_0 = e^{\tilde{\mathfrak{l}}_0} \subset G$ and let $\tilde{C}$ be the simple perverse sheaf on $\tilde{\mathfrak{l}}_\eta$ corresponding to A under the obvious isomorphism $\tilde{\mathfrak{l}}_\eta \xrightarrow{\sim} \mathfrak{l}_\eta$. Then $(\tilde{L}, \tilde{L}_0, \tilde{\mathfrak{l}}, \tilde{\mathfrak{l}}_*, \tilde{C})$ is an object of

$\mathfrak{T}_\eta$ and its $G_{\underline{0}}$-orbit is independent of the choice of splitting, by 2.7(a). Now let $(\mathfrak{p}'_*, L', P'_0, \mathfrak{l}', \mathfrak{l}'_*, \mathfrak{u}'_*) \in \mathfrak{P}^{\epsilon'}$, A' be as in 6.3 (with $\epsilon' = \epsilon''$) and assume that

$$^{\epsilon'}\widetilde{\mathrm{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(A') \cong B[d'] \oplus C',$$

where $d' \in \mathbf{Z}$ and $C' \in \mathcal{Q}_\eta^{\epsilon''}(\mathfrak{g}_\delta)$. We choose a splitting $\tilde{\mathfrak{l}}'_*$ of $\mathfrak{p}'_*$ and we associate to it a system $(\tilde{L}', \tilde{L}'_0, \tilde{\mathfrak{l}}', \tilde{\mathfrak{l}}'_*, \tilde{C}')$ just as $(\tilde{L}, \tilde{L}_0, \tilde{\mathfrak{l}}, \tilde{\mathfrak{l}}_*, \tilde{C})$ was defined in terms of $\tilde{\mathfrak{l}}$; here $\tilde{C}'$ corresponds to A'. Using 4.1(d), we see that

$$^{\epsilon'}\widetilde{\mathrm{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(D(A')) \cong D(B)[-d'] \oplus D(C').$$

Let Π be as in 6.4 (with A' replaced by $D(A')$ and $\epsilon' = \epsilon''$). From the definition of Π in 6.4 we have also

$$\Pi = \{B[d] \oplus C, D(B)[-d'] \oplus D(C')\} = v^{d-d'} \text{ plus an element in } \mathbf{N}((v)).$$

(We use 0.12.) In particular we have $\Pi \neq 0$ hence X' in 6.4 is nonempty. It follows that $(\tilde{L}', \tilde{L}'_0, \tilde{\mathfrak{l}}', \tilde{\mathfrak{l}}'_*, \tilde{C}')$ and $(\tilde{L}, \tilde{L}_0, \tilde{\mathfrak{l}}, \tilde{\mathfrak{l}}_*, \tilde{C})$ are in the same $G_{\underline{0}}$-orbit. This proves that $B \mapsto (\tilde{L}, \tilde{L}_0, \tilde{\mathfrak{l}}, \tilde{\mathfrak{l}}_*, \tilde{C})$ associates to B a well-defined element $\psi(B) \in \underline{\mathfrak{T}}_\eta$.

6.7. For any $\xi \in \underline{\mathfrak{T}}_\eta$ let $^\xi\mathcal{Q}_\eta^{\epsilon'}(\mathfrak{g}_\delta)$ be the full subcategory of $\mathcal{Q}_\eta^{\epsilon'}(\mathfrak{g}_\delta)$ whose objects are direct sums of shifts of simple perverse sheaves B in $\mathcal{Q}_\eta^{\epsilon'}(\mathfrak{g}_\delta)$ such that $\psi(B) = \xi$ (see 6.6); let $^\xi\mathcal{K}_\eta^{\epsilon'}(\mathfrak{g}_\delta)$ be the (free) $\mathcal{A}$-submodule of $\mathcal{K}_\eta^{\epsilon'}(\mathfrak{g}_\delta)$ with basis given by the simple perverse sheaves B in $^\xi\mathcal{Q}_\eta^{\epsilon'}(\mathfrak{g}_\delta)$. Clearly, we have

$$\mathcal{K}_\eta^{\epsilon'}(\mathfrak{g}_\delta) = \oplus_{\xi \in \underline{\mathfrak{T}}_\eta} {^\xi\mathcal{K}_\eta^{\epsilon'}}(\mathfrak{g}_\delta).$$

7. The categories $\mathcal{Q}(\mathfrak{g}_\delta)$, $\mathcal{Q}'(\mathfrak{g}_\delta)$

In this section we consider two categories of perverse sheaves $\mathcal{Q}(\mathfrak{g}_\delta)$, $\mathcal{Q}'(\mathfrak{g}_\delta)$ defined in terms of spiral induction; see 7.8. The simple objects in $\mathcal{Q}(\mathfrak{g}_\delta)$ are supported on $\mathfrak{g}_\delta^{nil}$, while those in $\mathcal{Q}'(\mathfrak{g}_\delta)$ have Fourier-Deligne transforms supported on $\mathfrak{g}_\delta^{nil}$. We also complete the proof of the main theorem 0.6.

7.1. Let $(\mathcal{O}, \mathcal{L}) \in \mathcal{I}(\mathfrak{g}_\delta)$. Let A_1 be the simple perverse sheaf on $\mathfrak{g}_\delta$ such that $\mathrm{supp}(A_1)$ is the closure $\bar{\mathcal{O}}$ of $\mathcal{O}$ in $\mathfrak{g}_\delta$ and $A_1|_{\mathcal{O}} = \mathcal{L}[\dim \mathcal{O}]$.

Choose $x \in \mathcal{O}$ and $\phi \in J_\delta(x)$; define $\mathfrak{p}_*^x, \tilde{\mathfrak{l}}_*^\phi, \tilde{L}^\phi, P_0$ as in 2.9. Then $\mathcal{Q}(\tilde{\mathfrak{l}}_\eta^\phi)$ is defined in terms of $\tilde{\mathfrak{l}}_*^\phi, \tilde{L}^\phi$ and for any $A' \in \mathcal{Q}(\tilde{\mathfrak{l}}_\eta^\phi)$ we can consider

$$I(A') := {^{\dot\eta}}\mathrm{Ind}_{\mathfrak{p}_\eta^x}^{\mathfrak{g}_\delta}(A') \in \mathcal{Q}_\eta^{\dot\eta}(\mathfrak{g}_\delta);$$

see 4.1. We show:

(a) *If $A' \in \mathcal{Q}(\tilde{\mathfrak{l}}_\eta^\phi)$, then the support of $I(A')$ is contained in $\bar{\mathcal{O}}$.*

Let $y \in \mathfrak{g}_\delta$ be in the support of $I(A')$. We must show that $y \in \bar{\mathcal{O}}$. From the definition of $I(A')$, there exists $g \in G_{\underline{0}}$ and $z \in \mathfrak{p}_\eta^x$ such that $\mathrm{Ad}(g)(z) = y$. Since the support of $I(A')$ and $\bar{\mathcal{O}}$ are $G_{\underline{0}}$-invariant we may replace y by $\mathrm{Ad}(g^{-1})y$ hence we may assume that $y \in \mathfrak{p}_\eta^x$. Using 2.9(e), we see that $\mathfrak{p}_\eta^x$ is equal to the closure of the P_0-orbit of x in $\mathfrak{p}_\eta^x$, which is clearly contained in $\bar{\mathcal{O}}$. This proves (a).

Recall that $x \in \overset{\circ\phi}{\tilde{\mathfrak{l}}_\eta}$ (see 2.9(b)) hence $\overset{\circ\phi}{\tilde{\mathfrak{l}}_\eta} \subset \mathcal{O}$. By 2.9(c), $\mathcal{L}_1 := \mathcal{L}|_{\overset{\circ\phi}{\tilde{\mathfrak{l}}_\eta}}$ is an irreducible $\tilde{L}_0^\phi$-equivariant local system on $\overset{\circ\phi}{\tilde{\mathfrak{l}}_\eta}$. Let $\mathcal{L}_1^\sharp \in \mathcal{D}(\tilde{\mathfrak{l}}_\eta^\phi)$ be as in 0.11 and let $A = \mathcal{L}_1^\sharp[\dim \tilde{\mathfrak{l}}_\eta^\phi]$. We show:

(b) *$I(\mathcal{L}_1^\sharp)|_{\mathcal{O}}$ is $\mathcal{L}$.*

Let $E'_{\mathcal{O}}$ be the inverse image of $\mathcal{O}$ under $c : E' \to \mathfrak{g}_\delta$ (where c, E' are as in 4.1 with $\mathfrak{p}_* = \mathfrak{p}_*^x$, $\epsilon = \dot\eta$). From the definitions we see that it is enough to check that the map $c_{\mathcal{O}} : E'_{\mathcal{O}} \to \mathcal{O}$ (restriction of c) is bijective on $\mathbf{k}$-points. Since $G_{\underline{0}}$ acts naturally on both $E'_{\mathcal{O}}$ and $\mathcal{O}$ compatibly with c and the action on $\mathcal{O}$ is transitive, it suffices to check that $c^{-1}(x)$ is a single point, namely (P_0, x). Let $(gP_0, x) \in c^{-1}(x)$. We have $g \in G_{\underline{0}}$, $\mathrm{Ad}(g^{-1})x \in \mathfrak{p}_\eta^x$ hence $x \in \mathrm{Ad}(g)\mathfrak{p}_\eta^x$. From 2.9(d) we deduce that $g \in P_0$ hence $(gP_0, x) = (P_0, x)$. This proves (b).

We show:

(c) *$I(\mathcal{L}_1^\sharp)$ is isomorphic to $\oplus_{j=1}^r A_j[t_j]$, where $t_1 = -\dim \mathcal{O}$ and for any $j \geq 2$, A_j is a simple $G_{\underline{0}}$-equivariant perverse sheaf on $\mathfrak{g}_\delta$ with support contained in $\bar{\mathcal{O}} - \mathcal{O}$ and $t_j \in \mathbf{Z}$.*

This follows from the fact that $I(\mathcal{L}_1^\sharp)$ is a semisimple $G_{\underline{0}}$-equivariant perverse sheaf on $\mathfrak{g}_\delta$ (the decomposition theorem), taking into account (a),(b).

By 1.5(a) we can find a parabolic subalgebra $\mathfrak{q}$ of $\tilde{\mathfrak{l}}^\phi$, a Levi subalgebra $\mathfrak{m}$ of $\mathfrak{q}$ (with $\mathfrak{q}, \mathfrak{m}$ compatible with the $\mathbf{Z}$-grading of $\tilde{\mathfrak{l}}^\phi$) and a cuspidal $M_0 := e^{\mathfrak{m}_0}$-equivariant perverse sheaf C on $\mathfrak{m}_\eta$ such that some shift of A is a direct summand of $\mathrm{ind}_{\mathfrak{q}_\eta}^{\tilde{\mathfrak{l}}_\eta^\phi}(C)$. From the definition we have

(d) $$\Psi(\mathcal{O}, \mathcal{L}) = (M, M_0, \mathfrak{m}, \mathfrak{m}_*, C) \in \underline{\mathfrak{T}}_\eta,$$

where $M = e^{\mathfrak{m}}$; see 3.5.

For any $N \in \mathbf{Z}$ let $\hat{\mathfrak{p}}_N$ be the inverse image of $\mathfrak{q}_N$ under the obvious map $\mathfrak{p}_N \to \mathfrak{l}_N$. Then by 2.8(a), $\hat{\mathfrak{p}}_*$ is an $\dot\eta$-spiral and $\mathfrak{m}_*$ is a splitting of it, so that, by 4.2(a), we have

$$^{\dot\eta}\,\mathrm{Ind}_{\hat{\mathfrak{p}}_\eta}^{\mathfrak{g}_\delta}(C) = {}^{\dot\eta}\,\mathrm{Ind}_{\mathfrak{p}_\eta^\phi}^{\mathfrak{g}_\delta}(\mathrm{ind}_{\mathfrak{q}_\eta}^{\tilde{\mathfrak{l}}_\eta^\phi}(C)).$$

It follows that some shift of $^{\dot\eta}\,\mathrm{Ind}_{\mathfrak{p}_\eta^\phi}^{\mathfrak{g}_\delta}(A)$ is a direct summand of $^{\dot\eta}\,\mathrm{Ind}_{\hat{\mathfrak{p}}_\eta}^{\mathfrak{g}_\delta}(C)$ hence, using (c), we see that some shift of A_1 is a direct summand of $^{\dot\eta}\,\mathrm{Ind}_{\hat{\mathfrak{p}}_\eta}^{\mathfrak{g}_\delta}(C)$. In particular we have $A_1 \in \mathcal{Q}_\eta^{\dot\eta}(\mathfrak{g}_\delta)$ and $\psi(A_1) = (M, M_0, \mathfrak{m}, \mathfrak{m}_*, C) \in \underline{\mathfrak{T}}_\eta$; see 6.6 (with $\epsilon = \dot\eta$). Comparing with (d) we see that:

(e) $\psi(A_1) = \Psi(\mathcal{O}, \mathcal{L})$.

7.2. **Characterization of $\mathcal{Q}_\eta^{\dot\eta}(\mathfrak{g}_\delta)$ as orbital sheaves.** Let A' be a semisimple $G_{\underline{0}}$-equivariant complex on $\mathfrak{g}_\delta$. We show:

(a) *We have $A' \in \mathcal{Q}_\eta^{\dot\eta}(\mathfrak{g}_\delta)$ if and only if* $\mathrm{supp}(A') \subset \mathfrak{g}_\delta^{nil}$.

We can assume that A' is a simple perverse sheaf. If $\mathrm{supp}(A') \subset \mathfrak{g}_\delta^{nil}$, then we have $A' \in \mathcal{Q}_\eta^{\dot\eta}(\mathfrak{g}_\delta)$ by the arguments in 7.1. Conversely, assume that $A' \in \mathcal{Q}_\eta^{\dot\eta}(\mathfrak{g}_\delta)$. We can find $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*) \in \mathfrak{P}^{\dot\eta}$ and $A \in \mathcal{Q}(\mathfrak{l}_\eta)$ such that some shift of A' is a direct summand of $B := {}^{\dot\eta}\,\mathrm{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)$. To show that $\mathrm{supp}(A') \subset \mathfrak{g}_\delta^{nil}$ it is enough to show that $\mathrm{supp}(B) \subset \mathfrak{g}_\delta^{nil}$ or (with c, A_1 as in 4.1 with $\epsilon = \dot\eta$) that $\mathrm{supp}(c_! A_1) \subset \mathfrak{g}_\delta^{nil}$. This would follow if we can show that the image of c is contained in $\mathfrak{g}_\delta^{nil}$. By the definition of c it is enough to show that $\mathfrak{p}_\eta \subset \mathfrak{g}_\delta^{nil}$. This follows from 2.5(d) applied with $N = \eta$.

We now restate 7.1(e) as follows.

(b) *Let A' be a simple perverse sheaf in $\mathcal{Q}_\eta^{\dot\eta}(\mathfrak{g}_\delta)$ and let $(\mathcal{O}, \mathcal{L}) \in \mathcal{I}(\mathfrak{g}_\delta)$ be such that* $\mathrm{supp}(A') = \bar{\mathcal{O}}$ *and* $A'|_{\mathcal{O}} = \mathcal{L}[\dim \mathcal{O}]$. *Then* $\psi(A') = \Psi(\mathcal{O}, \mathcal{L})$. *(Notation of* 3.5 *and* 6.6 *with* $\epsilon = \dot\eta$.)

7.3. We now give another proof of the following statement (see also 3.8(f)):

(a) *The map* $\Psi : \mathcal{I}(\mathfrak{g}_\delta) \to \underline{\mathfrak{T}}_\eta$ *in* 3.5 *is surjective.*

Let $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, C)$ be an element of $\underline{\mathfrak{T}}_\eta$. We can find an $\dot\eta$-spiral $\mathfrak{p}_*$ such that $\mathfrak{m}_*$ is a splitting of $\mathfrak{p}_*$. By 6.5(a), we have ${}^{\dot\eta}\widetilde{\mathrm{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(C) \neq 0$, that is, there exists a simple perverse sheaf A' in $\mathcal{Q}_\eta^{\dot\eta}(\mathfrak{g}_\delta)$ such that some shift of A' is a direct summand of ${}^{\dot\eta}\widetilde{\mathrm{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(C)$. It follows that $\psi(A') = (M, M_0, \mathfrak{m}, \mathfrak{m}_*, C)$ hence, by 7.2(b), we have $\Psi(\mathcal{O}, \mathcal{L}) = (M, M_0, \mathfrak{m}, \mathfrak{m}_*, C)$ where $(\mathcal{O}, \mathcal{L})$ corresponds to A' as in 7.2(b). This proves (a).

7.4. Until the end of 7.7 we assume that $p > 0$. If E, E' are finite dimensional $\mathbf{k}$-vector space with a given perfect bilinear pairing $E \times E' \to \mathbf{k}$, then we have the Fourier-Deligne transform functor $\Phi : \mathcal{D}(E) \to \mathcal{D}(E')$ defined in terms of a fixed nontrivial character $\mathbf{F}_p \to \bar{\mathbf{Q}}_l^*$ as in [L4, 1.9].

7.5. **Fourier transform and spiral restriction.** Let $B \in \mathcal{D}(\mathfrak{g}_\delta)$; we denote by $\Phi_{\mathfrak{g}}(B) \in \mathcal{D}(\mathfrak{g}_{-\delta})$ the Fourier-Deligne transform of B with respect to the perfect pairing $\mathfrak{g}_\delta \times \mathfrak{g}_{-\delta} \to \mathbf{k}$ defined by $\langle , \rangle$.

Let $\epsilon' \in \{1, -1\}$. Let $(\mathfrak{p}'_*, L', P'_0, \mathfrak{l}', \mathfrak{l}'_*, \mathfrak{u}'_*) \in \mathfrak{P}^{\epsilon'}$ and let

$$R_\eta = {}^{\epsilon'}\mathrm{Res}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(B) \in \mathcal{D}(\mathfrak{l}'_\eta), \quad R_{-\eta} = {}^{\epsilon'}\mathrm{Res}_{\mathfrak{p}'_{-\eta}}^{\mathfrak{g}_{-\delta}}(\Phi_{\mathfrak{g}}(B)) \in \mathcal{D}(\mathfrak{l}'_{-\eta}).$$

Then

(a) *$R_{-\eta}$ is the Fourier-Deligne transform of R_η with respect to the perfect pairing $\mathfrak{l}_\eta \times \mathfrak{l}_{-\eta} \to \mathbf{k}$ defined by $\langle , \rangle$.*

The proof is almost the same as that of [L4, 10.2]. We omit it.

7.6. **Fourier transform and spiral induction.** Let $\epsilon' \in \{1, -1\}$. Let

$$(\mathfrak{p}'_*, L', P'_0, \mathfrak{l}', \mathfrak{l}'_*, \mathfrak{u}'_*) \in \mathfrak{P}^{\epsilon'}.$$

Let $A \in \mathcal{D}(\mathfrak{l}'_\eta)$ be a semisimple complex; we denote by $\Phi_{\mathfrak{l}'}(A) \in \mathcal{D}(\mathfrak{l}'_{-\eta})$ the Fourier-Deligne transform of A with respect to the perfect pairing $\mathfrak{l}'_\eta \times \mathfrak{l}'_{-\eta} \to \mathbf{k}$ defined by $\langle , \rangle$; note that $\Phi_{\mathfrak{l}'}(A)$ is a semisimple complex. Let

$$I_\eta = {}^{\epsilon'}\widetilde{\mathrm{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(A) \in \mathcal{D}(\mathfrak{g}_\delta),$$
$$I_{-\eta} = {}^{\epsilon'}\widetilde{\mathrm{Ind}}_{\mathfrak{p}'_{-\eta}}^{\mathfrak{g}_{-\delta}}(\Phi_{\mathfrak{l}'}(A)) \in \mathcal{D}(\mathfrak{g}_{-\delta}).$$

Then:

(a) *$I_{-\eta}$ is the Fourier-Deligne transform of I_η with respect to the perfect pairing $\mathfrak{g}_\delta \times \mathfrak{g}_{-\delta} \to \mathbf{k}$ defined by $\langle , \rangle$.*

The proof is almost the same as that of [L5, A2]. We omit it.

7.7. **Characterization of $\mathcal{Q}_\eta^{-\dot\eta}(\mathfrak{g}_\delta)$ as anti-orbital sheaves.** Let $B \in \mathcal{D}(\mathfrak{g}_\delta)$ be a semisimple complex; let $B' = \Phi_\mathfrak{g}(B) \in \mathcal{D}(\mathfrak{g}_{-\delta})$ be its Fourier-Deligne transform, as in 7.5. Note that B' is again a semisimple complex. We show:

(a) *We have* $B \in \mathcal{Q}_\eta^{-\dot\eta}(\mathfrak{g}_\delta)$ *if and only if* $\mathrm{supp}(B') \subset \mathfrak{g}_{-\delta}^{nil}$.

We can assume that B (and hence also B') is a simple perverse sheaf.

Assume first that $B \in \mathcal{Q}_\eta^{-\dot\eta}(\mathfrak{g}_\delta)$. We can find $(\mathfrak{p}'_*, L', P'_0, \mathfrak{l}', \mathfrak{l}'_*, \mathfrak{u}'_*) \in \mathfrak{P}^{-\dot\eta}$ and a cuspidal perverse sheaf C in $\mathcal{Q}(\mathfrak{l}'_\eta)$ such that some shift of B is a direct summand of ${}^{-\dot\eta}\widetilde{\mathrm{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(C)$. Using 7.6(a) we see that some shift of B' is a direct summand of ${}^{-\dot\eta}\widetilde{\mathrm{Ind}}_{\mathfrak{p}'_{-\eta}}^{\mathfrak{g}_{-\delta}}(C')$ where $C' = \Phi_{\mathfrak{l}'}(C) \in \mathcal{D}(\mathfrak{l}'_{-\eta})$ (notation of 7.6). By [L4, 10.6], C' is a cuspidal perverse sheaf in $\mathcal{Q}(\mathfrak{l}'_{-\eta})$. It follows that $B' \in \mathcal{Q}_{-\eta}^{-\dot\eta}(\mathfrak{g}_{-\delta})$. Using 7.2(a) (with η, δ replaced by $-\eta, -\delta$) we deduce that $\mathrm{supp}(B') \subset \mathfrak{g}_{-\delta}^{nil}$.

Conversely, assume that B is such that $\mathrm{supp}(B') \subset \mathfrak{g}_{-\delta}^{nil}$. Using 7.2(a), we see that $B' \in \mathcal{Q}_{-\eta}^{-\dot\eta}(\mathfrak{g}_{-\delta})$. We can find $(\mathfrak{p}'_*, L', P'_0, \mathfrak{l}', \mathfrak{l}'_*, \mathfrak{u}'_*) \in \mathfrak{P}^{-\dot\eta}$ and a cuspidal perverse sheaf C'_1 in $\mathcal{Q}(\mathfrak{l}'_{-\eta})$ such that some shift of B' is a direct summand of ${}^{-\dot\eta}\widetilde{\mathrm{Ind}}_{\mathfrak{p}'_{-\eta}}^{\mathfrak{g}_{-\delta}}(C'_1)$. We can find a cuspidal perverse sheaf C_1 in $\mathcal{Q}(\mathfrak{l}'_\eta)$ such that $C'_1 = \Phi_{\mathfrak{l}'}(C)$ (we use again [L4, 10.6]). Using 7.6(a), we see that some shift of $\Phi_\mathfrak{g}(B)$ is a direct summand of $\Phi_\mathfrak{g}({}^{-\dot\eta}\widetilde{\mathrm{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(C_1))$ hence some shift of B is a direct summand of ${}^{-\dot\eta}\widetilde{\mathrm{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(C_1)$ so that $B \in \mathcal{Q}_\eta^{-\dot\eta}(\mathfrak{g}_\delta)$. This completes the proof of (a).

7.8. The assumption on p in 7.4 is no longer in force. From 7.2(a) we see that $\mathcal{Q}_\eta^{\dot\eta}(\mathfrak{g}_\delta)$ (hence also $\mathcal{K}_\eta^{\dot\eta}(\mathfrak{g}_\delta)$) is independent of η as long as $\underline{\eta} = \delta$. We shall write $\mathcal{Q}(\mathfrak{g}_\delta), \mathcal{K}(\mathfrak{g}_\delta)$ instead of $\mathcal{Q}_\eta^{\dot\eta}(\mathfrak{g}_\delta)$, $\mathcal{K}_\eta^{\dot\eta}(\mathfrak{g}_\delta)$. From 7.7(a) we see that $\mathcal{Q}_\eta^{-\dot\eta}(\mathfrak{g}_\delta)$ (hence also $\mathcal{K}_\eta^{-\dot\eta}(\mathfrak{g}_\delta)$) is independent of η as long as $\underline{\eta} = \delta$ (at least when $p > 0$, but then the same holds for $p = 0$ by standard arguments). We shall write $\mathcal{Q}'(\mathfrak{g}_\delta), \mathcal{K}'(\mathfrak{g}_\delta)$ instead of $\mathcal{Q}_\eta^{-\dot\eta}(\mathfrak{g}_\delta)$, $\mathcal{K}_\eta^{-\dot\eta}(\mathfrak{g}_\delta)$.

For $\xi \in \underline{\mathfrak{T}}_\delta$ we write ${}^\xi\mathcal{Q}(\mathfrak{g}_\delta), {}^\xi\mathcal{K}(\mathfrak{g}_\delta)$ instead of ${}^\xi\mathcal{Q}_\eta^{\dot\eta}(\mathfrak{g}_\delta)$, ${}^\xi\mathcal{K}_\eta^{\dot\eta}(\mathfrak{g}_\delta)$ and we write ${}^\xi\mathcal{Q}'(\mathfrak{g}_\delta), {}^\xi\mathcal{K}'(\mathfrak{g}_\delta)$ instead of ${}^\xi\mathcal{Q}_\eta^{-\dot\eta}(\mathfrak{g}_\delta)$, ${}^\xi\mathcal{K}_\eta^{-\dot\eta}(\mathfrak{g}_\delta)$. The discussion in 3.9 shows that ${}^\xi\mathcal{Q}(\mathfrak{g}_\delta), {}^\xi\mathcal{K}(\mathfrak{g}_\delta)$ and ${}^\xi\mathcal{Q}'(\mathfrak{g}_\delta), {}^\xi\mathcal{K}'(\mathfrak{g}_\delta)$ are independent of η as long as $\underline{\eta} = \delta$.

7.9. **Proof of Theorem 0.6.** Let $\xi \in \underline{\mathfrak{T}}_\eta$. Let $K \in \mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_\delta^{nil})$. We say that $K \in \mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_\delta^{nil})_\xi$ if any simple perverse sheaf B which appears in a perverse cohomology sheaf of K satisfies $\psi(B) = \xi$; note that B belongs to $\mathcal{Q}_\eta^{\dot\eta}(\mathfrak{g}_\delta)$, see 7.2(a); hence $\psi(B)$ is defined as in 6.6.

Now let ξ, ξ' in $\underline{\mathfrak{T}}_\eta$ be such that $\xi \neq \xi'$. Let $K \in \mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_\delta^{nil})_\xi$, $K' \in \mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_\delta^{nil})_{\xi'}$. We show:

(a) $\mathrm{Hom}_{\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_\delta^{nil})}(K, K') = 0$.

We can assume that $K = B[n], K' = B'[n']$ where B, B' are simple perverse sheaves in $\mathcal{Q}_\eta^{\dot\eta}(\mathfrak{g}_\delta)$ such that $\psi(B) = \xi, \psi(B') = \xi'$ and n, n' are integers. We see that it is enough to prove (a) in the case where $K = {}^{\dot\eta}\widetilde{\mathrm{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(A')[n]$, $K' = {}^{\dot\eta}\widetilde{\mathrm{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)[n']$ with $n, n' \in \mathbf{Z}$, $\mathfrak{p}_*, \mathfrak{p}'_*, A, A'$ as in 6.4, and $\epsilon' = \epsilon'' = \dot\eta$, since some shifts of B and B' appear as direct summands of such K and K'. By 0.12(a), we have an isomorphism

$$\mathrm{Hom}_{\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_\delta^{nil})}(K, K') = \mathbf{D}_0(\mathfrak{g}_\delta^{nil}, G_{\underline{0}}; K, D(K'))^*.$$

Hence

(b) $\dim \mathrm{Hom}_{\mathcal{D}_{G_{\underline{0}}}(\mathfrak{g}_\delta^{nil})}(K, K') = d_{n-n'}(\mathfrak{g}_\delta^{nil}; {}^{\dot\eta}\widetilde{\mathrm{Ind}}_{\mathfrak{p}'_\eta}^{\mathfrak{g}_\delta}(A'), {}^{\dot\eta}\widetilde{\mathrm{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(D(A)))$.

Here we use 4.1(d). Since $\xi \neq \xi'$, the set X' defined in 6.4 for the pair $(D(A), A')$ is empty. Therefore the right side of (b) is zero by 6.4. Then (a) follows from (b). We see that Theorem 0.6 holds.

8. Monomial and quasi-monomial objects

The results in this section are parallel to those in 1.8–1.9. They serve as preparation for the next section.

8.1. Let $\epsilon = \dot{\eta}$. We denote by $\mathfrak{R}^\epsilon$ the set of all data of the form

$$(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*, A),$$

where $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*) \in \mathfrak{P}^\epsilon$ (see 4.1) and A is a perverse sheaf in $\mathcal{Q}(\mathfrak{l}_\eta)$ which is η-semicuspidal (as in 1.8 with H replaced by L).

8.2. An object $B \in \mathcal{Q}(\mathfrak{g}_\delta)$ is said to be *η-quasi-monomial* if $B \cong {}^\epsilon\widetilde{\mathrm{Ind}}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A)$ for some $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*, A) \in \mathfrak{R}^\epsilon$; if in addition A is taken to be cuspidal, then B is said to be *η-monomial.* Using 1.8(b) and the transitivity property 4.2, we see that:

(a) *If $B \in \mathcal{Q}(\mathfrak{g}_\delta)$ is η-quasi-monomial, then there exists an η-monomial object $B' \in \mathcal{Q}(\mathfrak{g}_\delta)$ such that $B' \cong B[a_1] \oplus B[a_2] \oplus \cdots \oplus B[a_k]$ for some sequence $a_1, a_2, \ldots, a_k$ in $\mathbf{Z}$, $k \geq 1$. In particular, in $\mathcal{K}(\mathfrak{g}_\delta)$ we have $(B') = (v^{a_1} + \cdots + v^{a_k})(B)$.*

An object of $\mathcal{Q}(\mathfrak{g}_\delta)$ is said to be *η-good* if it is a direct sum of shifts of η-quasi-monomial objects.

Proposition 8.3 (8.3)**.** *Let $B \in \mathcal{Q}(\mathfrak{g}_\delta)$. There exists η-good objects B_1, B_2 in $\mathcal{Q}(\mathfrak{g}_\delta)$ such that $B \oplus B_1 \cong B_2$.*

We can assume that B is a simple perverse sheaf. We define $(\mathcal{O}, \mathcal{L}) \in \mathcal{I}(\mathfrak{g}_\delta)$ by the requirement that $\mathrm{supp}\, B$ is the closure $\bar{\mathcal{O}}$ of $\mathcal{O}$ in $\mathfrak{g}_\delta$ and $B|_{\mathcal{O}} = \mathcal{L}[\dim \mathcal{O}]$. We prove the proposition by induction on $\dim \mathcal{O}$. Let $x \in \mathcal{O}$. We associate to x an ϵ-spiral $\mathfrak{p}_* = \mathfrak{p}_*^x$ as in 2.9; we complete it uniquely to a system $(\mathfrak{p}_*, L, P_0, \mathfrak{l}, \mathfrak{l}_*, \mathfrak{u}_*) \in \mathfrak{P}^\epsilon$. By 7.1(c), there exists $A_1 \in \mathcal{Q}(\mathfrak{l}_\eta)$ such that ${}^\epsilon\,\mathrm{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A_1) \cong B[d] \oplus B'$, where $d \in \mathbf{Z}$ and $B' \in \mathcal{Q}(\mathfrak{g}_\delta)$ has support contained in $\bar{\mathcal{O}} - \mathcal{O}$. We now use 1.9(a) for L, A_1 instead of H, A_1; applying ${}^\epsilon\,\mathrm{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}$ to the equality in 1.9(a) we obtain

$${}^\epsilon\,\mathrm{Ind}_{\mathfrak{p}_\eta}^{\mathfrak{g}_\delta}(A_1) \oplus C'_1 \oplus C'_2 \oplus \ldots \oplus C'_t = C'_{t+1} \oplus \ldots \oplus C'_{t+t'},$$

where each C'_j is an η-quasi-monomial object with a shift (we have used the transitivity property 4.2). Thus we have

$$B[d] \oplus B' \oplus C'_1 \oplus C'_2 \oplus \ldots \oplus C'_t = C'_{t+1} \oplus \ldots \oplus C'_{t+t'}.$$

Now the induction hypothesis implies that B' is η-good. From this and the previous equality we see that B is η-good. The proposition is proved.

Corollary 8.4.

(a) *The $\mathcal{A}$-module $\mathcal{K}(\mathfrak{g}_\delta)$ is generated by the classes of η-quasi-monomial objects of $\mathcal{Q}(\mathfrak{g}_\delta)$.*

(b) *The $\mathbf{Q}(v)$-vector space $\mathbf{Q}(v) \otimes_{\mathcal{A}} \mathcal{K}(\mathfrak{g}_\delta)$ is generated by the classes of η-monomial objects of $\mathcal{Q}(\mathfrak{g}_\delta)$.*

(a) follows immediately from 8.3; (b) follows from (a) using 8.2(a).

8.5. We show:

(a) *If* B_1, B_2 *are elements of* $\mathcal{K}(\mathfrak{g}_\delta)$ *then* $\{B_1, B_2\} \in \mathbf{Q}(v)$ *(notation of* 4.4(c)*).*

By 8.3, we can assume that B_1, B_2 are classes of η-quasi-monomial objects. By 8.2(a) we have $f_1 B_1 = B'_1$, $f_2 B_2 = B'_2$ where B'_1, B'_2 represent ϵ-monomial objects and f_1, f_2 are nonzero elements of $\mathcal{A}$. Thus, we can assume that B_1, B_2 represent η-monomial objects. In this case the result follows from 6.4.

9. Examples

In this section we consider examples where $G = SL(V)$ or $Sp(V)$. We assume that $m \geq 2$ and $\eta = 1$ hence $\delta = \underline{1}$. We write "spiral" instead of "1-spiral". We explicitly describe the spirals and the set of blocks $\underline{\mathfrak{T}}_1$ in both cases, and describe the map Ψ in the case $G = SL(V)$.

9.1. **Spirals for the cyclic quiver.** We preserve the notation from 0.3. Thus we assume that $G = SL(V)$ where $V = \oplus_{i \in \mathbf{Z}/m} V_i$. We have an induced $\mathbf{Z}/m$-grading on $\mathfrak{g} = \mathfrak{sl}(V)$, so that $\mathfrak{g}_{\underline{1}}$ is the space of all maps in 0.3(a). In general, we have $\mathfrak{g}_i = \oplus_{j \in \mathbf{Z}/m} \operatorname{Hom}(V_j, V_{j+i})$.

The datum $\lambda \in Y_{G_{\underline{0}}, \mathbf{Q}}$ is the same as a $\mathbf{Q}$-grading on each V_i, i.e., $V_i = \oplus_{x \in \mathbf{Q}} ({}_x V_i)$ such that $\sum_i \sum_x x \dim({}_x V_i) = 0$. Given such a $\mathbf{Q}$-grading on each V_i, the corresponding spiral $\mathfrak{p}_* = \{\mathfrak{p}_N \subset \mathfrak{g}_{\underline{N}}\}_{N \in \mathbf{Z}}$ takes the following form:

$$\mathfrak{p}_N = \{\phi \in \mathfrak{sl}(V) | \phi({}_x V_j) \subset \oplus_{x' \geq x+N} ({}_{x'} V_{j+\underline{N}}), \quad \forall j \in \mathbf{Z}/m, x \in \mathbf{Q}\}.$$

A splitting $\mathfrak{m}_* = \{\mathfrak{m}_N \subset \mathfrak{g}_{\underline{N}}\}_{N \in \mathbf{Z}}$ of the spiral $\mathfrak{p}_*$ takes the form

$$\mathfrak{m}_N = \{\phi \in \mathfrak{sl}(V) | \phi({}_x V_j) \subset {}_{x+N} V_{j+\underline{N}}, \quad \forall j \in \mathbf{Z}/m, x \in \mathbf{Q}\}.$$

For such a grading ${}_x V_i$ we may introduce a quiver Q_λ as follows. Let J_λ be the finite set of pairs $(i, x) \in \mathbf{Z}/m \times \mathbf{Q}$ such that ${}_x V_i \neq 0$. Then Q_λ has vertex set J_λ and an edge $(i, x) \to (i+1, x+1)$ if both (i, x) and $(i+1, x+1)$ are in J_λ. Then Q_λ is a disjoint union of directed chains (that is, quivers of type A with exactly one source and exactly one sink). We may identify $\mathfrak{m}_1$ with the representation space of the quiver Q_λ with vector space ${}_x V_i$ on the vertex $(i, x) \in J_\lambda$.

Let B be the set of chains in Q_λ, and let $J_\lambda = \sqcup_{\beta \in B} ({}_\beta J_\lambda)$ be the corresponding decomposition of the vertex set. Let ${}_\beta V := \oplus_{(i,x) \in \beta} ({}_x V_i)$. Then we have $V = \oplus_{\beta \in B} ({}_\beta V)$. Let $M = e^{\mathfrak{m}}, M_0 = e^{\mathfrak{m}_0}$ where $\mathfrak{m} = \oplus_N \mathfrak{m}_N$. Then $M = S(\prod_{\beta \in B} GL({}_\beta V))$, $M_0 = S(\prod_{(i,x) \in J_\lambda} GL({}_x V_i))$. The center Z_M is the subgroup of M where each factor in $GL({}_\beta V)$ is a scalar matrix.

9.2. **Admissible systems for the cyclic quiver.** Let d be a divisor of $n = \dim V$. Suppose that the following hold:

(1) Each ${}_x V_i$ has dimension ≤ 1.

(2) Each connected component of the quiver Q_λ is a directed chain containing exactly d vertices.

In this case, M_0 is a maximal torus of G stabilizing each line ${}_x V_i$ for $(i, x) \in J_\lambda$. The open M_0-orbit $\mathring{\mathfrak{m}}_1 \subset \mathfrak{m}_1$ consists of representations of Q_λ where all arrows are nonzero (hence isomorphisms). The stabilizer of an element in $\mathring{\mathfrak{m}}_1$ under M_0 is exactly Z_M, which acts by a scalar z_β on each chain $\beta \in B$, such that $(\prod_{\beta \in B} z_\beta)^d = 1$. We see that $\pi_0(Z_M) \cong \mu_d$. For any primitive character $\chi : \mu_d \to \bar{\mathbf{Q}}_l^*$, we have a rank 1 M_0-equivariant local system C_χ on $\overset{\circ}{\mathfrak{m}}_1$ on whose stalks Z_M acts via χ. This is a cuspidal local system because it is the restriction of the cuspidal local system on the regular nilpotent orbit of $\mathfrak{m}$ with central character χ. Let $\tilde{C}_\chi$ be the cuspidal perverse sheaf on $\mathfrak{m}_1$ corresponding to C_χ. The system $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}_\chi)$ is admissible. It is easy to see that any admissible system is of the form we just described.

Given such a grading λ, we define a function $f : B \to \mathbf{Z}/m$ such that $f(\beta) = i$ where (i, x) is the head (origin) of the chain β. Each vertex $(i, x) \in J_\lambda$ lies in a unique chain $\beta \in B$ whose head is of the form $(f(\beta), x')$. Then $x - x' = y$ is an integer between 0 and $d-1$ and $f(\beta) + \underline{y} = i$ in $\mathbf{Z}/m$. This implies that $\dim V_i = \sharp\{x \in \mathbf{Q} | (i, x) \in J_\lambda\}$ is the same as the number of pairs $(\beta, y) \in B \times \{0, 1, \ldots, d-1\}$ such that $f(\beta) + \underline{y} = i$. Choosing a bijection between $\{1, 2, \ldots, n/d\}$ and B, the function f may be viewed as a function $\{1, 2, \ldots, n/d\} \to \mathbf{Z}/m$ satisfying 0.7(b). Changing the bijection amounts to precomposing f with a permutation of $\{1, 2, \ldots, n/d\}$. Summarizing the above discussion, we get a canonical bijection between $\underline{\mathfrak{T}}_1$ and the set of equivalence classes of triples (d, f, χ) as in 0.7(a).

9.3. **The map Ψ for the cyclic quiver.** We preserve the notation from 9.1. Let $(\mathcal{O}, \mathcal{L}) \in \chi(i\mathfrak{g}_{\underline{1}})$. For each element $e \in \mathcal{O}$, there exists a decomposition of V into Jordan blocks $\{{}_\alpha W\}_{\alpha \in B_e}$ compatible with the $\mathbf{Z}/m$-grading in the following sense. Each Jordan block ${}_\alpha W$ is a direct sum of finitely many 1-dimensional subspaces indexed by $0, 1, \ldots$, i.e., ${}_\alpha W = ({}_\alpha W_0) \oplus ({}_\alpha W_1) \oplus \ldots$ such that

(1) ${}_\alpha W_N \subset V_{h(\alpha)+\underline{N}}$ for some $h(\alpha) \in \mathbf{Z}/m$ (location of the head of the Jordan block α);

(2) e maps ${}_\alpha W_N$ isomorphically to ${}_\alpha W_{N+1}$ whenever $N \geq 0$ and ${}_\alpha W_{N+1} \neq 0$.

The datum of $\{{}_\alpha W\}_{\alpha \in B_e}$ as above is the equivalent to the datum of an element $\phi \in J_{\underline{1}}(e)$; see 2.3. From this we may define a quiver Q_e whose vertex set J_e consists of pairs $(\alpha, N) \in B_e \times \mathbf{Z}_{\geq 0}$ such that ${}_\alpha W_N \neq 0$, and there is no edge $(\alpha, N) \to (\alpha, N+1)$ if both $(\alpha, N), (\alpha, N+1)$ are in $B_e \times \mathbf{Z}_{\geq 0}$.

Each vertex (α, N) is labelled with the element $h(\alpha) + \underline{N} \in \mathbf{Z}/m$. The isomorphism class of Q_e together with the labelling by elements in $\mathbf{Z}/m$ is independent of the choice of e in $\mathcal{O}$ and the choice of the Jordan block decomposition. Therefore we denote this labelled quiver by $Q_\mathcal{O}$, with vertex set $J_\mathcal{O}$ and set of chains $B_\mathcal{O}$.

Let $d' = \gcd\{|\alpha|\}_{\alpha \in B_\mathcal{O}}$ (here $|\alpha|$ is the number of vertices of the chain α). Then for any $e \in \mathcal{O}$, there is a canonical isomorphism $\pi_0(G_{\underline{0}}(e)) \cong \mu_{d'}$. The local system $\mathcal{L}$ on $\mathcal{O}$ corresponds to a character ρ of $\mu_{d'}$, which has order d dividing d' and a unique factorization

$$\rho : \mu_{d'} \to \mu_d \xrightarrow{\chi} \bar{\mathbf{Q}}_l^*$$

such that χ is injective (here the first map $\mu_{d'} \to \mu_d$ is given by $z \mapsto z^{d'/d}$). Now we define a new quiver $Q_\mathcal{O}^{[d]}$ by removing certain edges from each chain of $Q_\mathcal{O}$ such that each chain of $Q_\mathcal{O}^{[d]}$ has exactly d vertices. Let B be the set of chains of $Q_\mathcal{O}^{[d]}$; then B can be identified with the set $\{1, 2, \ldots, n/d\}$. Define $f : \{1, 2, \ldots, n/d\} \cong B \to \mathbf{Z}/m$ to be the map assigning to each $\beta \in B$ the label of its head. This way we get a triple (d, f, χ) as in 0.7(b) whose equivalence class is well-defined.

Proposition 9.4. *In the case of cyclic quivers, the map $\Psi : \mathcal{I}(\mathfrak{g}_{\underline{1}}) \to \underline{\mathfrak{T}}_1$ sends $(\mathcal{O}, \mathcal{L})$ to the admissible system in $\underline{\mathfrak{T}}_1$ which corresponds to the equivalence class of the triple (d, f, χ) defined above under the bijection* 0.7(a).

Let $e \in \mathcal{O}$, and let $V = \oplus_{\alpha \in B_e}({}_\alpha W)$, ${}_\alpha W = {}_\alpha W_0 \oplus {}_\alpha W_1 \oplus \cdots$ be a Jordan block decomposition, where ${}_\alpha W_N \subset V_{h(\alpha)+\underline{N}}$ for $\alpha \in B_e, N \in \mathbf{Z}_{\geq 0}$. Let L be the Levi subgroup of a parabolic subgroup of G such that L stabilizes the decomposition $V = \oplus_{\alpha \in B_e}({}_\alpha W)$. Then $\mathfrak{l} = \mathfrak{L}L$ has a $\mathbf{Z}$-grading induced from the $\mathbf{Z}$-grading on each of ${}_\alpha W$. In particular, $\mathfrak{l}_1$ is the space of representations of the quiver Q_e. The system $(L, L_0, \mathfrak{l}, \mathfrak{l}_*)$ is the system $(\tilde{L}^\phi, \tilde{L}_0^\phi, \tilde{\mathfrak{l}}^\phi, \tilde{\mathfrak{l}}_*^\phi)$ attached to some $\phi \in J_{\underline{1}}(e)$ as in 2.9. Then e is in the open L_0-orbit $\mathring{\mathfrak{l}}_1$ of $\mathfrak{l}_1$, which is contained in the regular nilpotent orbit of $\mathfrak{l}$.

Let ${}_\alpha L = SL({}_\alpha W)$ be the subgroup of L which acts as identity on all blocks ${}_{\alpha'} W$ for $\alpha' \neq \alpha$. Then ${}_\alpha \mathfrak{l} = \mathfrak{L}({}_\alpha L)$ carries a $\mathbf{Z}$-grading compatible with that on $\mathfrak{l}$. For each interval $[a, b] \subset \mathbf{Z}_{\geq 0}$, let ${}_\alpha W_{[a,b]} \subset {}_\alpha W$ be the direct sum of ${}_\alpha W_N$ for $a \leq N \leq b$. We decompose ${}_\alpha W$ into $|\alpha|/d$ parts each of dimension d:

(a) $${}_\alpha W = \oplus_{j=1}^{|\alpha|/d}({}_\alpha W_{[(j-1)d, jd-1]}).$$

Let ${}_\alpha M \subset {}_\alpha L$ be the subgroup stabilizing the decomposition (a). Then the Lie algebra ${}_\alpha \mathfrak{m}$ of ${}_\alpha M$ inherits a $\mathbf{Z}$-grading from that of ${}_\alpha \mathfrak{l}$, and the open orbit ${}_\alpha \mathring{\mathfrak{m}}_1$ carries a local system ${}_\alpha C_\chi$ corresponding to the character χ of $\mu_d \cong \pi_0(Z({}_\alpha M))$. Let ${}_\alpha \tilde{C}_\chi$ be the cuspidal perverse sheaf on ${}_\alpha \mathfrak{m}_1$ corresponding to ${}_\alpha C_\chi$. Define a parabolic subalgebra ${}_\alpha \mathfrak{q} \subset {}_\alpha \mathfrak{l}$ to be the stabilizer of the filtration ${}_\alpha W_{[|\alpha|-d, |\alpha|-1]} \subset {}_\alpha W_{[|\alpha|-2d, |\alpha|-1]} \subset \cdots \subset {}_\alpha W = {}_\alpha W_{[0, |\alpha|-1]}$. Then ${}_\alpha \mathfrak{q}$ is compatible with the $\mathbf{Z}$-grading on ${}_\alpha \mathfrak{l}$ and ${}_\alpha \mathfrak{m}$ is a Levi subalgebra of ${}_\alpha \mathfrak{q}$. The induction

$$\mathrm{ind}_{{}_\alpha \mathfrak{q}_1}^{{}_\alpha \mathfrak{l}_1}({}_\alpha \tilde{C}_\chi)$$

restricted to ${}_\alpha \mathring{\mathfrak{l}}_1$ is isomorphic to $\mathcal{L}|_{{}_\alpha \mathring{\mathfrak{l}}_1}$, because the map c in 1.3 (applied to ${}_\alpha \mathfrak{l}, {}_\alpha \mathfrak{q}, {}_\alpha \mathfrak{m}$ in place of $\mathfrak{h}, \mathfrak{p}, \mathfrak{l}$) is an isomorphism when restricted to ${}_\alpha \mathring{\mathfrak{l}}_1$. Therefore the middle extension of $\mathcal{L}|_{{}_\alpha \mathring{\mathfrak{l}}_1}$ to $\mathfrak{l}_1$ appears as a direct summand of $\mathrm{ind}_{{}_\alpha \mathfrak{q}_1}^{{}_\alpha \mathfrak{l}_1}({}_\alpha \tilde{C}_\chi)$. Therefore, under the map defined in 1.5(b), the image of $({}_\alpha \mathring{\mathfrak{l}}_1, \mathcal{L}|_{{}_\alpha \mathring{\mathfrak{l}}_1})$ is

$$({}_\alpha M, {}_\alpha M_0, {}_\alpha \mathfrak{m}, {}_\alpha \mathfrak{m}_1, {}_\alpha \tilde{C}_\chi).$$

Let ${}_\alpha \tilde{M} \subset GL({}_\alpha W)$ be the stabilizer of the decomposition (a). Let

$$M = S(\prod_{\alpha \in B_e}({}_\alpha \tilde{M})) \subset L$$

with Lie algebra $\mathfrak{m} \subset \oplus({}_\alpha \tilde{\mathfrak{m}})$ and the induced $\mathbf{Z}$-grading from each ${}_\alpha \tilde{\mathfrak{m}} = \mathfrak{L}({}_\alpha \tilde{M})$. The open M_0-orbit on $\mathfrak{m}_1 = \oplus({}_\alpha \mathfrak{m}_1)$ is $\mathring{\mathfrak{m}}_1 = \prod({}_\alpha \mathring{\mathfrak{m}}_1)$. Let $C_\chi = \boxtimes({}_\alpha C_\chi)$ on $\mathring{\mathfrak{m}}_1$. Let $\tilde{C}_\chi$ be the cuspidal perverse sheaf on $\mathfrak{m}_1$ corresponding to C_χ. By the compatibility of the assignment in 1.5(b) with direct products, in the situation $H = L$, the pair $(\mathring{\mathfrak{l}}_1, \mathcal{L}|_{\mathring{\mathfrak{l}}_1})$ maps to $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}_\chi)$. Therefore, $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}_\chi)$ is the admissible system attached to $(\mathcal{O}, \mathcal{L})$ through the procedure in 2.9. By 9.2, the admissible system $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C}_\chi)$ corresponds to the triple (d, f, χ) defined in 9.3 before the statement of this proposition. This finishes the proof.

9.5. **The symplectic quiver.** Let V be a finite-dimensional vector space over $\mathbf{k}$ with a nondegenerate symplectic form ω. Assume that m in 0.1 is even. Let $\tilde{\mathfrak{S}}_m = \{j; j = k/2; k = \text{ an odd integer}\}$ and let $\mathfrak{S}_m$ be the set of equivalence classes for the relation $\sim$ on $\tilde{\mathfrak{S}}_m$ given by $j \sim j'$ if $j - j' \in m\mathbf{Z}$. Note that the involution $j \mapsto -j$ of $\tilde{\mathfrak{S}}_m$ induces an involution of $\mathfrak{S}_m$ denoted again by $j \mapsto -j$.

For any $N \in \mathbf{Z}$, the map $j \mapsto N + j$ of $\tilde{\mathfrak{S}}_m$ onto itself induces a map of $\mathfrak{S}_m$ onto itself which depends only on $\underline{N}$ and is denoted by $j \mapsto \underline{N} + j$.

The set $\mathfrak{S}_m$ consists of m elements represented by

$$\{\frac{1}{2}, \frac{3}{2}, \dots, \frac{m-1}{2}, \frac{m+1}{2}, \dots, m - \frac{1}{2}\}.$$

Consider a grading on V indexed by $\mathfrak{S}_m$:

(a) $$V = \bigoplus_{j \in \mathfrak{S}_m} V_j,$$

such that $\omega(V_j, V_{j'}) = 0$ unless $j' = -j$ (as elements of $\mathfrak{S}_m$). Using the symplectic form, for $j \in \mathfrak{S}_m$ we may identify V_j with the dual of V_{-j}.

We assume that $G = Sp(V)$ and that the $\mathbf{Z}/m$-grading of $\mathfrak{g} = \mathfrak{sp}(V)$ is given by

(b) $$\mathfrak{g}_i = \{\phi \in \mathfrak{sp}(V) | \phi(V_j) \subset V_{i+j}, \quad \forall j \in \mathfrak{S}_m\}, \quad \forall i \in \mathbf{Z}/m.$$

In particular, an element $\phi \in \mathfrak{g}_{\underline{1}}$ is a collection of maps $\phi_i : V_{i-\frac{1}{2}} \to V_{i+\frac{1}{2}}$, $i \in \mathbf{Z}/m$, which can be represented by a cyclic quiver

$$\begin{array}{ccccccc}
V_{-\frac{1}{2}} & \xleftarrow{\phi_{-1}} & V_{-\frac{3}{2}} & \longleftarrow & \dots & \xleftarrow{\phi_{\frac{m}{2}+1}} & V_{\frac{m+1}{2}} \\
\phi_0 \downarrow & & & & & & \uparrow \phi_{\frac{m}{2}} \\
V_{\frac{1}{2}} & \xrightarrow{\phi_1} & V_{\frac{3}{2}} & \longrightarrow & \dots & \xrightarrow{\phi_{\frac{m}{2}-1}} & V_{\frac{m-1}{2}}
\end{array}.$$

The condition $\phi \in \mathfrak{sp}(V)$ becomes that

(c) $$\phi_{-i} = -\phi_i^*, \quad \forall i \in \mathbf{Z}/m.$$

Here $\phi_i^* : V_{i+\frac{1}{2}}^* \to V_{i-\frac{1}{2}}^*$ is the adjoint of ϕ_i, which can be viewed as a map $V_{-i-\frac{1}{2}} \to V_{-i+\frac{1}{2}}$ under the identifications $V_{i+\frac{1}{2}}^* \cong V_{-i-\frac{1}{2}}$, $V_{i-\frac{1}{2}}^* \cong V_{-i+\frac{1}{2}}$ using the symplectic pairing. In particular, for $i = 0$, $\phi_0 : V_{-\frac{1}{2}} = V_{\frac{1}{2}}^* \to V_{\frac{1}{2}}$ can be viewed as a vector $\phi_0 \in V_{\frac{1}{2}}^{\otimes 2}$. The condition (b) for $i = 0$ is equivalent to saying that $\phi_0 \in \mathrm{Sym}^2(V_{\frac{1}{2}})$. Similarly, we may view $\phi_{\frac{m}{2}}$ as a vector in $V_{\frac{m+1}{2}}^{\otimes 2}$, and the condition (c) for $i = \frac{m}{2}$ is equivalent to saying that $\phi_{\frac{m}{2}} \in \mathrm{Sym}^2(V_{\frac{m+1}{2}})$.

We call a representation of the quiver above in which $V_{-j} = V_j^*$, and (c) holds a *symplectic representation*. In other words, $\mathfrak{g}_{\underline{1}}$ is the space of symplectic representations of the quiver above.

We have $G_{\underline{0}} \cong \prod_{\frac{1}{2} \le j \le \frac{m-1}{2}} GL(V_j)$, where $GL(V_j) \cong GL(V_{-j})$ acts diagonally on both V_j and $V_{-j} = V_{m-j} = V_j^*$.

9.6. **Spirals for the symplectic quiver.** Each element $\lambda \in Y_{G_{\underline{0}},\mathbf{Q}}$ is the same datum as a $\mathbf{Q}$-grading on each V_j, $j \in \mathfrak{S}_m$, i.e., $V_j = \oplus_{x \in \mathbf{Q}} ({}_xV_j)$ such that under the symplectic form ω, $\omega({}_xV_j, {}_{x'}V_{-j}) = 0$ unless $x + x' = 0$. Then ${}_{-x}V_{-j}$ can be identified with the dual of ${}_xV_j$ for all $(j, x) \in \mathfrak{S}_m \times \mathbf{Q}$. The spiral $\mathfrak{p}_*$ associated to this grading is

$$\mathfrak{p}_N = \{\phi \in \mathfrak{sp}(V) | \phi({}_xV_j) \subset \oplus_{x' \ge x+N} ({}_{x'}V_{j+\underline{N}}), \quad \forall j \in \mathfrak{S}_m, x \in \mathbf{Q}\}.$$

A splitting $\mathfrak{m}_*$ of the spiral $\mathfrak{p}_*$ takes the form

$$\mathfrak{m}_N = \{\pi \in \mathfrak{sp}(V) | \phi({}_xV_j) \subset {}_{x+N}V_{j+\underline{N}}, \quad \forall j \in \mathfrak{S}_m, x \in \mathbf{Q}\}.$$

To each such grading, we may attach a quiver Q_λ as we did for the cyclic quiver (since the symplectic quiver is a special case of a cyclic quiver). There is an involution on Q_λ sending $(j, x) \in J_\lambda$ to $(-j, -x) \in J_\lambda$. This involution stabilizes at most two chains Q'_λ and Q''_λ of Q_λ. The set of vertices of Q'_λ (possibly empty) is $J'_\lambda := \{(x, x) | {}_xV_x \neq 0\} \subset J_\lambda$. The set of vertices of Q''_λ (possibly empty) is $J''_\lambda := \{(x - \frac{m}{2}, x) | {}_xV_{x-\frac{m}{2}} \neq 0\} \subset J_\lambda$.

9.7. **Admissible systems for the symplectic quiver.** Suppose that the following hold:

(1) For each $(j, x) \in J - (J'_\lambda \sqcup J''_\lambda)$, we have $\dim {}_xV_j = 1$.

(2) The chains in Q_λ other than Q'_λ and Q''_λ all consist of a single vertex.

(3) Let $\sharp J'_\lambda = 2a'$ for some $a' \in \mathbf{Z}_{\geq 0}$. When $a' > 0$, $(-a' + \frac{1}{2}, -a' + \frac{1}{2})$ is the head of J'_λ and $(a' - \frac{1}{2}, a' - \frac{1}{2})$ is the tail. Then $\dim {}_xV_x = a' + \frac{1}{2} - |x|$ for all $(x, x) \in J'_\lambda$.

(4) Let $\sharp J''_\lambda = 2a''$ for some $a'' \in \mathbf{Z}_{\geq 0}$. When $a'' > 0$, $(-a'' - \frac{m-1}{2}, -a'' + \frac{1}{2})$ is the head of J''_λ and $(a'' - \frac{m+1}{2}, a'' - \frac{1}{2})$ is the tail. Then $\dim {}_xV_{x-\frac{m}{2}} = a'' + \frac{1}{2} - |x|$ for all $(x - \frac{m}{2}, x) \in J''_\lambda$.

Under these assumptions, $\mathfrak{m}_1 = \mathfrak{m}'_1 \oplus \mathfrak{m}''_1$, where $\mathfrak{m}'_1$ is the space of representations of the quiver Q'_λ with dimension vector $\dim {}_xV_x = a' + \frac{1}{2} - |x|$ and satisfying the duality condition $\psi_i = -\psi^*_{-i}$ (where $\psi_i : {}_{i-\frac{1}{2}}V_{i-\frac{1}{2}} \to {}_{i+\frac{1}{2}}V_{i+\frac{1}{2}}$) for all $i \in \{-a' + 1, \dots, a' - 1\}$. Similarly, $\mathfrak{m}''_1$ is the space of representations of the quiver Q''_λ with dimension vector $\dim {}_xV_{x-\frac{m}{2}} = a'' + \frac{1}{2} - |x|$ and satisfying the duality condition $\psi_i = -\psi^*_{-i}$. The open M_0-orbit $\mathring{\mathfrak{m}}_1$ consists of those representations of Q'_λ and Q''_λ where each arrow has maximal rank (either injective or surjective).

Let $V' = \oplus_x {}_xV_x$ and $V'' = \oplus_x {}_xV_{x-\frac{m}{2}}$. Let $V^\dagger = \oplus_{(j,x) \notin J'_\lambda \cup J''_\lambda} ({}_xV_j)$. Then we have $V = V' \oplus V'' \oplus V^\dagger$. This decomposition is preserved by M, and $M \cong Sp(V') \times Sp(V'') \times T^\dagger$, where $T^\dagger$ is the maximal torus in $Sp(V^\dagger)$ stabilizing each line ${}_xV_j \subset V^\dagger$. The center Z_M is isomorphic to $\{\pm 1\} \times \{\pm 1\} \times T^\dagger$ under this decomposition. The stabilizer of a point in $\mathring{\mathfrak{m}}_1$ under M_0 is exactly Z_M. Let C be the rank one local system on $\mathring{\mathfrak{m}}_1$ on whose stalks $\pi_0(Z_M)$ acts nontrivially on both factors of $\{\pm 1\}$. Then C is cuspidal because it is the restriction of the unique cuspidal local system on $\mathfrak{m}$. Let $\tilde{C}$ be the cuspidal perverse sheaf on $\mathfrak{m}_1$ defined by C. The system $(M, M_0, \mathfrak{m}, \mathfrak{m}_*, \tilde{C})$ is admissible. Moreover, any admissible system is of this form. Under $G_{\underline{0}}$-conjugacy, the only invariant of an admissible system is given by the numbers a' and a''. Since $\dim V'_j + \dim V''_j \leq \dim V_j$, we have the following inequality for all $j \in \mathfrak{S}_m$:

(a)
$$\begin{aligned} \dim V_j \geq\ & \sharp\{-a' + \frac{1}{2} \leq x \leq a' - \frac{1}{2} | x \equiv j \mod m\mathbf{Z}\} \\ & + \sharp\{-a'' + \frac{1}{2} \leq x \leq a'' - \frac{1}{2} | x \equiv j + \frac{m}{2} \mod m\mathbf{Z}\}. \end{aligned}$$

To summarize, we have a natural bijection

(b) $$\underline{\mathfrak{T}}_1 \leftrightarrow \{(a', a'') \in \mathbf{Z}_{\geq 0} \times \mathbf{Z}_{\geq 0} \text{ satisfying (a) for all } j \in \mathfrak{S}_m\}.$$

The map $\Psi : \mathcal{I}(\mathfrak{g}_{\underline{1}}) \to \underline{\mathfrak{T}}_1$ for the symplectic quiver as well as other graded Lie algebras of classical type will be described in a sequel to this paper using the combinatorics of symbols.

References

[KL] David Kazhdan and George Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras*, Invent. Math. **87** (1987), no. 1, 153–215, DOI 10.1007/BF01389157. MR862716

[Ko] Bertram Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math. **81** (1959), 973–1032, DOI 10.2307/2372999. MR0114875

[L1] G. Lusztig, *Intersection cohomology complexes on a reductive group*, Invent. Math. **75** (1984), no. 2, 205–272, DOI 10.1007/BF01388564. MR732546

[L2] George Lusztig, *Character sheaves. I*, Adv. in Math. **56** (1985), no. 3, 193–237, DOI 10.1016/0001-8708(85)90034-9. MR792706

[L3] George Lusztig, *Character sheaves. II*, Adv. in Math. **57** (1985), no. 3, 266–315, DOI 10.1016/0001-8708(85)90064-7. MR806210

[L4] George Lusztig, *Study of perverse sheaves arising from graded Lie algebras*, Adv. Math. **112** (1995), no. 2, 147–217, DOI 10.1006/aima.1995.1031. MR1327095

[RR] L. Rider and A. Russell, *Perverse sheaves on the nilpotent cone and Lusztig's generalized Springer correspondence*, arxiv:1409.7132.

[L5] G. Lusztig, *Study of antiorbital complexes*, Representation theory and mathematical physics, Contemp. Math., vol. 557, Amer. Math. Soc., Providence, RI, 2011, pp. 259–287, DOI 10.1090/conm/557/11036. MR2848930

[St] Robert Steinberg, *Endomorphisms of linear algebraic groups*, Memoirs of the American Mathematical Society, No. 80, American Mathematical Society, Providence, R.I., 1968. MR0230728

[Vi] È. B. Vinberg, *The Weyl group of a graded Lie algebra* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), no. 3, 488–526, 709. MR0430168

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Masssachusetts 02139

E-mail address: `gyuri@math.mit.edu`

Department of Mathematics, Yale University, New Haven, Connecticut 06511

E-mail address: `zhiweiyun@gmail.com`